Izvestiya, Atmospheric and Oceanic Physics

Vol. 26, No. 1, 1990
UDC 551.511.3

General Form of the Equations of Atmospheric Dynamics in Isobaric Coordinates

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The equations of motion of an ideal atmosphere are converted from Cartesian coordinates to isobaric coordinates with no assumptions of quasistatic behavior. In contrast to the quasistatic equations, these equations can also describe acoustic waves. It is shown that for infinitely slow vertical movements, the system is converted to the quasistatic equations. Invariant forms of the equations for the energy and potential vorticity in isobaric coordinates are derived.

1. Introduction. Eliassen's suggestion [1] that the equations of atmospheric dynamics be written in isobaric coordinates has proved very fruitful. Isobaric coordinates and the closely associated sigma-system [2] are now widely used for numerical modeling of atmospheric dynamics.

Hitherto, in transforming the dynamic equations into isobaric coordinates it has been assumed that the atmosphere is in quasistatic equilibrium and that one of the equations of motion can be replaced by the static equation

$$\partial p/\partial z = -g\rho, \tag{1}$$

where p is the pressure, ρ is the density, g is the gravitional acceleration, and z is the height above sea level.

In the present paper we shall show that condition (1) is not an essential (necessary) condition. The conversion can be made under the far less confining condition of a monotonic change of pressure with height,

$$\partial p/\partial z < 0. \tag{2}$$

There are, of course, processes in which this condition breaks down, e.g., the propagation of shock waves, subsonic flow, and the like. Such processes cannot be realistically described in isobaric coordinates. But in most problems that are of interest in atmospheric physics, condition (2) is satisfied everywhere and at all times.

The present paper is methodological in character. We shall investigate primarily the problem of invariant transformation of the equations of motion into isobaric coordinates. Using the general apparatus of tensor analysis in a frame of reference in curvilinear motion, we can represent all characteristics of motion and the dynamic equations in invariant form. The quasistatic equations will be derived for the special case of infinitely slow vertical movement.

It is obvious that the quasistatic approximation will continue to be used in most practical problems. But with the more general equations available, when necessary we can use isobaric coordinates to take account of subtler effects associated with deviations from static equilibrium. In addition, this approach enables us to investigate more deeply the structure of the isobaric coordinate space and the invariant properties of the equations of motion.

2. The Isobaric System of Coordinates. In a plane-parallel atmosphere we introduce the rectangular coordinates (x, y, z) and the associated orthogonal basis $\{\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z\}$ (where $\mathbf{i}_z$ is directed upward). Together with this system, we consider the system of coordinates in which the third coordinate is the pressure p. We denote the system as $\{x^1, x^2, x^3\}$. The conversion from the old coordinates to the new isobaric coordinates consists of

$$x^1 = x, \quad x^2 = y, \quad x^3 = p(x, y, z, t), \tag{3}$$

where $p(x, y, z, t)$ is the pressure field in atmosphere. For this conversion to be single-valued and for there to exist the reverse transformation

$$x = x^1, \quad y = x^2, \quad z = z(x^1, x^2, x^3, t), \tag{4}$$

condition (2) must be satisfied.

Below we shall present, very briefly, the main vector equations that will be needed for the discussion [3-6] in curvilinear coordinates $\{x^1, x^2, x^3\}$.

The covariant and contravariant bases $\{\mathbf{e}^\alpha\}$ and $\{\mathbf{e}_\alpha\}$ have, in the original orthogonal basis $\{\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z\}$, the representations

$$\mathbf{e}^1 = \mathbf{i}_x, \quad \mathbf{e}^2 = \mathbf{i}_y, \quad \mathbf{e}^3 = \nabla p, \tag{5a}$$

$$\mathbf{e}_1 = \mathbf{i}_x + \mathbf{i}_z z_{,1}, \quad \mathbf{e}_2 = \mathbf{i}_y + \mathbf{i}_z z_{,2}, \quad \mathbf{e}_3 = \mathbf{i}_z z_{,3}, \tag{5b}$$

0001-4338/91/2601-0002$18.00/1

where ∇ is the gradient operator (here, in Castesian coordinates), and $z_{,\alpha}=\partial z/\partial x^{\alpha}$. The spatial locations of the basis vectors relative to the isobaric surface $z=z(x, y, p, t)$, p = const, are shown in the figure.

For the covariant and contravariant components a_{α} and a^{α} of the vector $\mathbf{a}=a_{\alpha}\mathbf{e}^{\alpha}=a^{\alpha}\mathbf{e}_{\alpha}$, the equations

$$a^1=a_x, \quad a^2=a_y, \quad a^3=\mathbf{a}\nabla p, \tag{6a}$$

$$a_1=a_x+a_z z_{,1}, \quad a_2=a_y+a_z z_{,2}, \quad a_3=a_z z_{,3}, \tag{6b}$$

apply, where a_x, a_y, and a_z are the components of vector $\mathbf{a}$ in basis $\{\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z\}$.

Bases (5a) and (5b) can also be used to express the components of the metric tensor G:

$$G^{\alpha\beta}=\mathbf{e}^{\alpha}\mathbf{e}^{\beta}, \quad G_{\alpha\beta}=\mathbf{e}_{\alpha}\mathbf{e}_{\beta}.$$

In the formulas that follow, a fundamental role is played by the determinant of the metric tensor,

$$G\equiv\|G_{\alpha\beta}\|=(z_{,3})^2=(\partial z/\partial p)^2. \tag{7}$$

3. Velocity. To represent the velocity of a material particle

$$\mathbf{v}=\mathbf{i}_x\frac{dx}{dt}+\mathbf{i}_y\frac{dy}{dt}+\mathbf{i}_z\frac{dz}{dt}=\mathbf{i}_x\dot{x}+\mathbf{i}_y\dot{y}+\mathbf{i}_z\dot{z} \tag{8}$$

in isobaric coordinates, it is convenient to use Eqs. (6). For the contravariant components, we find

$$v^1=\dot{x}, \quad v^2=\dot{y}, \quad v^3=\mathbf{v}\nabla p=\dot{p}-p_{,t}.$$

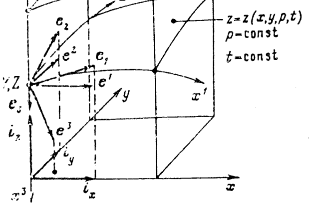
It is readily shown that

$$(p, t)_{x,y,z}=-(z_{,p})^{-1}_{x,y,t}(z_{,t})_{x,y,p}.$$

and, consequently,

$$v^1=\dot{x}, \quad v^2=\dot{y}, \quad v^3=\dot{p}+z_{,t}/z_{,p}, \tag{9}$$

in vector form,



Local bases in isobaric coordinates.

$$\mathbf{v}=\mathbf{u}+\mathbf{v}_0, \tag{10}$$

where

$$\mathbf{u}=u^{\alpha}\mathbf{e}_{\alpha}=\dot{x}\mathbf{e}_1+\dot{y}\mathbf{e}_2+\dot{p}\mathbf{e}_3 \tag{11}$$

is the velocity of the material point relative to the isobaric frame of reference, and

$$\mathbf{v}_0=\mathbf{e}_3 z_{,t}/z_{,p}=\mathbf{i}_z z_{,t} \tag{12}$$

is the speed of vertical displacement of the isobaric surface. Equations (9)-(12) form a contravariant description of the velocity.

But we often must also know the covariant description. From formula (6b) we find

$$v_1=\dot{x}+\dot{z}z_{,1}, \quad v_2=\dot{y}+\dot{z}z_{,2}, \quad v_3=\dot{z}z_{,3}. \tag{13}$$

Writing this result in vector form, we obtain

$$\mathbf{v}=\dot{x}\mathbf{e}^1+\dot{y}\mathbf{e}^2+\dot{z}\nabla z, \tag{14}$$

where $\nabla z=\mathbf{e}^{\alpha}z_{,\alpha}$ is the gradient of the function $z(x^i, t)$ in isobaric coordinates. Since

$$\nabla z=\mathbf{i}_z, \tag{15}$$

Eq. (14) actually coincides with the definition of velocity in Cartesian coordinates.

4. Density and Continuity Equations. Defining the density ρ and ρ_p by means of the equations

$$dm=\rho dxdydz=\rho_p dx^1dx^2dx^3,$$

where dm is the elementary mass associated with the volume $dV = dxdydz$, we obtain

$$\rho_p=\sqrt{G}\rho=-z_{,3}\rho. \tag{16}$$

The continuity equation (the law of conservation of mass) is

$$\rho_{,t}+\mathrm{div}(\rho\mathbf{v})=0, \tag{17a}$$

or

$$\dot{\rho}+\rho\,\mathrm{div}\,\mathbf{v}=0. \tag{17b}$$

If we decompose $\mathbf{v}$ into the relative velocity $\mathbf{u}$ and the velocity of the frame of reference $\mathbf{v}_0$ (see Eqs. (10)-(12)), then use for the divergence of the vector the invariant definition

$$\mathrm{div}\,\mathbf{a}=\frac{1}{\sqrt{G}}(\sqrt{G}a^{\alpha})_{,\alpha}, \tag{18}$$

and make use of the readily demonstrated equation

$$\mathrm{div}\,\mathbf{v}_0=\frac{1}{\sqrt{G}}\frac{\partial}{\partial t}\sqrt{G}. \tag{19}$$

Equations (17) can be written

$$\rho_{p,t} + (\rho_p u^\alpha)_{,\alpha} = 0, \qquad (20a)$$

$$\rho_p + \rho_p u^\alpha_{,\alpha} = 0. \qquad (20b)$$

As we should expect, the form of continuity equations (20) is retained in isobaric coordinates. Note that Eq. (20) contains the relative velocity $\{u^\alpha\} = \{\dot{x}, \dot{y}, \dot{p}\}$ rather than the absolute velocity.

5. Equations of Motion. The equations of motion in isobaric coordinates are readily derived from the appropriate equations

$$\ddot{x} = F_x, \quad \ddot{y} = F_y, \quad \ddot{z} = F_z \qquad (21)$$

in Cartesian coordinates. Here, the left sides of the equations contain the absolute acceleration of the material particle and the right sides contain the components of the force acting on this particle per unit mass.

In converting to isobaric coordinates, the absolute accelerations are not altered, but only their interpretations: $\dot{x}$, $\dot{y}$ must be regarded as the function $z(x^i, t)$ defining the position of the curvilinear frame of reference in space.

The right sides of Eqs. (21) can be rewritten, using Eqs. (5a) and (15), as

$$F_x = \mathbf{i}_x\mathbf{F} = \mathbf{e}^1\mathbf{F} = F^1,$$
$$F_y = \mathbf{i}_y\mathbf{F} = \mathbf{e}^2\mathbf{F} = F^2,$$
$$F_z = \mathbf{i}_z\mathbf{F} = (\nabla z)\,\mathbf{F} = F^\alpha z_{,\alpha}.$$

Thus, in isobaric coordinates the equations of motion are

$$\dot{u}^1 = F^1, \quad \dot{u}^2 = F^2, \quad \ddot{z} = F^\alpha z_{,\alpha}, \qquad (22)$$

where the operator for full differentiation with respect to time in isobaric coordinates is

$$\dot{f} \equiv df/dt = f_{,t} + u^\alpha f_{,\alpha}, \qquad (23)$$

and F^α are contravariant components of the force field vector. The problem of representing the equations of motion in isobaric coordinates will be fully solved if we represent the explicit equations for the force components F^α.

6. Force in Isobaric Coordinates. Consider an atmosphere rotating along with the earth under the ideal-gas approximation. Then the force in Cartesian coordinates (rotating along with the earth) will be

$$\mathbf{F} = -g\mathbf{i}_z - \frac{1}{\rho}\nabla p - \mathbf{F}_c, \qquad (24)$$

where $-\mathbf{F}_c$ is the Coriolis force,

$$\mathbf{F}_c = 2\boldsymbol{\omega}\times\mathbf{v}, \qquad (25)$$

and ω is the angular velocity vector of the earth's rotation,

$$\boldsymbol{\omega} = \mathbf{i}_y\omega_y(y) + \mathbf{i}_z\omega_z(y). \qquad (26)$$

Thus,

$$\mathbf{e}^\alpha\mathbf{F} = -g\mathbf{e}^\alpha\mathbf{i}_z - \frac{1}{\rho}\mathbf{e}^\alpha\nabla p - \mathbf{e}^\alpha\mathbf{F}_c, \quad \alpha = 1, 2,$$
$$\mathbf{F}\nabla z = -g\mathbf{i}_z\nabla z - \frac{1}{\rho}\nabla z\nabla p - \mathbf{F}_c\nabla z.$$

From Eqs. (15) and (16) and the definition of vector $\mathbf{e}^3$ (5a), we obtain

$$\mathbf{e}^\alpha\mathbf{F} = -\frac{\sqrt{G}}{\rho_p}\mathbf{e}^\alpha\mathbf{e}^3 - \mathbf{e}^\alpha\mathbf{E}_c, \quad \alpha = 1, 2,$$
$$\mathbf{F}\nabla z = -g - \frac{\sqrt{G}}{\rho_p}\mathbf{i}_z\nabla p - \mathbf{F}_c\mathbf{i}_z.$$

It is readily shown that

$$\mathbf{e}^\alpha\mathbf{e}^3 = G^{\alpha 3} = z_{,\alpha}/\sqrt{G}, \quad \alpha = 1, 2,$$
$$\mathbf{i}_z\nabla p = -1/\sqrt{G}.$$

and thus that

$$\mathbf{e}^\alpha\mathbf{F} = -z_{,\alpha}/\rho_p - \mathbf{e}^\alpha\mathbf{F}_c, \quad \alpha = 1, 2,$$
$$\mathbf{F}\nabla z = 1/\rho_p - g - \mathbf{F}_c\mathbf{i}_z. \qquad (27)$$

It remains to represent the Coriolis force in terms of the components of vectors $\boldsymbol{\omega}$ and $\mathbf{v}$. For this purpose it is convenient to use the representation of these vectors in Cartesian coordinates (e.g., Eqs. (8) and (26)). Using the definition of the vector product in the orthogonal system and the qualities $\dot{x} = u^1$, $\dot{y} = u^2$, we obtain

$$\mathbf{e}^1\mathbf{F}_c = \mathbf{i}_x\mathbf{F}_c = -2\omega_z u^2 + 2\omega_y\dot{z},$$
$$\mathbf{e}^2\mathbf{F}_c = \mathbf{i}_y\mathbf{F}_c = 2\omega_z u^1,$$
$$\mathbf{i}_z\mathbf{F}_c = -2\omega_y u^1. \qquad (28)$$

7. Closed System of Equations for the Hydrothermodynamics of the Atmosphere in Isobaric Coordinates. To obtain a closed system of equations describing the evolution of the atmosphere, we need to supplement equations of motion (22) with continuity equation (17), an equation for the balance of internal energy, and an equation of state. In addition, for a closed dynamic description, we need to add boundary value conditions to this system. As we shall shortly explain, one of the boundary conditions is converted in isobaric coordinates to a supplementary dynamic equation.

The internal energy equation undergoes no outward alteration on conversion to isobaric coordinates, but in the equation of state the density ρ is replaced by ρ_p, using Eq. (16). Thus, making use of the results of the preceding section, we obtain the system

$$\ddot{z} = 1/\rho_p - g + 2\omega_y u^1, \qquad (29a)$$

$$\dot{u}^1 = -z_{,1}/\rho_p + 2\omega_z u^2 - 2\omega_y \dot{z}, \qquad (29b)$$

$$\dot{u}^2 = -z_{,2}/\rho_p - 2\omega_z u^1, \qquad (29c)$$

$$c_p\dot{T} - RTu^3/p = w, \quad (29d)$$

$$\dot{\rho}_p + \rho_p u^{\alpha}_{,\alpha} = 0, \quad (29e)$$

$$z_{,p} = -R\rho_p T/p. \quad (29f)$$

Here $T(x^i, t) = T(x, y, p, t)$ is the temperature field, w is the density of the heat sources, and c_p and R are the thermodynamic constants of an ideal gas.

System (29) must be supplemented by boundary conditions on the underlying surface $z = 0$ and at $z = \infty$. The level $z = 0$ in isobaric coordinates represents the free surface $p=p_0(x, y, t)$, defined by the conditions

$$z[x, y, p_0(x, y, t), t] = 0. \quad (30)$$

The boundary condition for this surface is

$$\dot{z}(x, y, p, t)|_{p=p_0(x,y,t)} = 0. \quad (31)$$

Applying the operator of full differentiation with respect to time to Eq. (30), by virtue of Eq. (31) we obtain

$$\dot{p}_0 - u^3|_{p=p_0} = 0. \quad (32)$$

This equation, which defines the movement of the free surface in isobaric coordinates, is the equivalent of boundary value condition (31) for the case in which condition (30) is satisfied at the initial instant. Thus, the boundary value condition for the underlying surface becomes a dynamic equation in isobaric coordinates.

We integrate Eq. (29a) and use condition (30) to determine the constant of integration:

$$z(x, y, p, t) = R\int_p^{p_0(x,y,t)} \frac{\rho_p T}{p'}\, dp'. \quad (33)$$

This integral equation can be used in place of differential equation (29f); condition (30) is satisfied at all times.

It follows from Eq. (33) that the equation $\dot{z}|_{p=0}=0$ cannot be used as a boundary value condition at infinity. To see that this is so, consider, for example, the model situation

$$T = T_0 + T_1(t)\,\delta(p - p_1), \quad 0 < p_1 \leqslant p_0.$$

The correct boundary condition as $p \to 0$ is

$$u^3|_{p=0} = 0. \quad (34)$$

The components of the Coriolis force that depend on the horizontal components of the angular velocity are small compared with the other forces in Eqs. (29a) and (29b), i.e., $|2\omega_y u^1| \sim 10^{-4} g$, $|2\omega_y \dot{z}| \sim 10^{-4} |2\omega_z u^1|$, for slow quasihorizontal movements of the atmosphere, and we can therefore use in place of Eqs. (29a)-(29c) the equations

$$\ddot{z} = 1/\rho_p - g, \quad (35a)$$

$$\dot{u}^1 = -z_{,1}/\rho_p + 2\omega_z u^2. \quad (35b)$$

$$\dot{u}^2 = -z_{,2}/\rho_p - 2\omega_z u^1. \quad (35c)$$

8. Conversion to the Quasistatic Case. For slow quasihorizontal movements of the atmosphere, the vertical acceration $\ddot{z}$ is small compared with the forces ρ_p^{-1} and g (it is well known that $|\ddot{z}| \sim 10^{-4} g$ for movements of synoptic scale). Thus, when describing such movements, we can convert to the quasistatic equations. We derive the quasistatic equations by replacing $\ddot{z}$ with with zero in Eq. (35a), which is then converted to the condition of quasistatic equilibrium

$$\rho_p = 1/g. \quad (36)$$

Thus, the principal sign of quasistatic movement is that the density ρ_p is constant. Based on Eq. (29a), we obtain instead of (36)

$$\rho_p = 1/(g - 2\omega_y u^1),$$

i.e., with allowance for the Coriolis force, the density ρ_p depends on the horizontal velocity u_1. But this relationship is very weak, and the relative deviation of ρ_p from the constant value of Eq. (36) caused by the Coriolis force does not exceed 10^{-4}.

When condition (36) is satisfied, continuity equation (29f) is converted to the incompressibility condition

$$u^{\alpha}_{,\alpha} = 0, \quad (37)$$

and Eq. (33) to the familiar formula for the geopotential

$$gz(x, y, p, t) = R\int_p^{p_0(x,y,t)} \frac{T(x, y, p', t)}{p'}\, dp'. \quad (38)$$

Thus, under the assumption of infinitely slow vertical movements, the general system of equations in isobaric coordinates is converted to the usual, familiar system of quasistatic equations.

A special case of the quasistatic situation is the state of static equilibrium

$$u=0,\ z=z_0(p),\ T=T_0(p),\ p_0=a\approx 1000 \text{ mbar}. \quad (39)$$

In static equilibrium, the system of equations for the hydrothermodynamics of the atmosphere degenerates into the two main equations (36) and (38), with (38) becoming

$$gz_0(p) = R\int_p^a \frac{T_0(p')}{p'}\, dp'. \quad (40)$$

9. Linearized Equations. Linearizing the equations of hydrothermodynamics relative to the state of static equilibrium (36), (39), (40) results in the following system (we use as initial equations (29d), (29e), (32), (33) and (35a)-(35c)):

$$z'_{,tt} - u^3_{,t} c_0^2(p)/gp = -g^2 \rho'_p, \quad (41a)$$

$$u'_{,t} = -g z'_{,1} + 2\omega_z u^2, \quad (41b)$$

$$u^2_{,t} = -g z'_{,2} - 2\omega_z u^1, \quad (41c)$$

$$T'_{,t} - u^3 c_1^2(p)/(Rp) = 0, \quad (41d)$$

$$\rho'_{p,t} + u^{\alpha}_{,\alpha}/g = 0, \quad (41e)$$

$$p'_{0,t} - u^3|_{p=a} = 0, \quad (41f)$$

$$z' = \frac{c_0^2(0)}{ag} p'_0 + \int_p^a \frac{c_0^2(p')}{p'} \rho'_p dp' + \frac{R}{g} \int_p^a \frac{T'}{p'} dp'_0. \quad (41g)$$

Here, $z'=z(x, y, p, t)-z_0(p)$, $\rho_p'=\rho_p(x, y, p, t)-1/g$, $T'=T(x, y, p, t)-T_0(p)$, $p_0'(x, y, t)=p_0(x, y, t)-a$, are the deviations of the height of the isobaric surface, the density, temperature, and pressure at the underlying surface from their equilibrium values. The quantities

$$c_0(p) = \sqrt{RT_0(p)}, \; c_1(p) = \sqrt{RT_0\left(\frac{R}{c_p} - \frac{p}{T_0}\frac{\partial T_0}{\partial p}\right)} \quad (42)$$

have the dimensions of velocity. In the troposphere they are $c_0 \approx 280$ m/s, $c_1 \approx 180$ m/s. In the quasistatic approximation, c_0 and c_1 define the phase velocities of the propagation of external two-dimensional waves and internal gravity waves [7-9].

The solutions of Eqs. (14) also described wave movements of the atmosphere. But in contrast to the quasistatic situation, in this case the solutions contain not only gravity waves and gyroscopic waves, but also acoustic waves. For example, there exist solutions with a purely vertical structure (such solutions cannot exist in quasistatics):

$$u^1 = u^2 = 0,$$

$$u^3 \sim e^{i\omega_k t}\left[\left(\frac{p}{a}\right)^{1/2+ik} - \left(\frac{p}{a}\right)^{1/2-ik}\right], \quad (43)$$

where the frequency is given by the formula

$$\omega_k = g\sqrt{\frac{k^2+1/4}{c_0^2 - c_1^2}}, \quad (44)$$

k is the wave number (we assume that c_0 and c_1 are constants). Solution (43) is a superposition of two traveling waves, propagating vertically upward and downward in the atmosphere, which in Cartesian coordinates have the phase velocity

$$v_F = c_0\sqrt{\frac{1+1/(4k^2)}{1-c_1^2/c_0^2}}.$$

In the short-wave part of the spectrum ($k \gg 1$),

$$c_0 \leqslant v_F \leqslant c = \sqrt{\frac{c_p}{c_V} RT_0} \quad (c_V = c_p - R).$$

The equality on the left applies in the case of adiabatic equilibrium, and that on the right in an isothermal atmosphere. Here, c is the speed of sound in an ideal gas. Thus, Eq. (43) describes an acoustic standing wave.

10. Balance of Energy and Potential Vorticity. In conclusion, we consider two balance equations, which, in the absence of dissipative processes, heat sources, and external forces, become conservation laws, namely, the equations for balance of energy and potential vorticity.

The law of balance of energy in Cartesian coordinates is

$$\rho\dot{e} + \operatorname{div}(pv) = \rho w. \quad (45)$$

The mass energy density is the sum of the kinetic, potential, and internal energies,

$$e = v^2/2 + gz + c_v T. \quad (46)$$

Using Eqs. (10), (18), and (19), we can convert (45) into

$$\rho_p \dot{e}_p + (zu^{\alpha})_{,\alpha} = \rho_p w, \quad (47)$$

where

$$e_p = e + RT - z/\rho_n = {}^1/_2[(u^1)^2 + (u^2)^2 + (\dot{z})^2] + (g - 1/\rho_p)z + c_p T, \quad (48)$$

is the sum of the densities of the kinetic energy, the enthalpy, and the energy $(g-1/\rho_p)z$, resulting from the deviation of the density ρ_p from the equilibrium value $1/g$.

For the quasistatic case $\rho_p = 1/g$, $\dot{z}=0$, Eq. (47) becomes

$$\dot{e}_{kc} + (gzu^{\alpha})_{,\alpha} = w,$$

where $e_{kc} = {}^1/_2[(u^1)^2 + (u^2)^2] + c_p T$ is the energy density of the quasistatic atmosphere in isobaric coordinates.

Equations (46) and (48) define two different energy densities (which, incidentally, have the same dimensionality of energy/mass). The question arises whether there correspond to these two densities two different integral quantities that are conserved. We find that this is not the case. Integration in the vertical direction gives the same values for the energy of a unit column

$$\int_0^\infty e\rho dz = \int_0^{p_0} e_p \rho_p dp.$$

The Ertel potential vorticity J [10-12] is defined as

$$J = (\Omega \nabla s)/\rho \tag{49}$$

where $\Omega = \text{rot}\, \mathbf{v} + 2\omega$ is the absolute vorticity and $s = c_p \ln T - R \ln p$ + const is the entropy density of an ideal gas. From the equations of motion follows the balance of vorticity in Cartesian coordinates

$$\dot{J} = \frac{1}{\rho} \text{div}\left(\Omega \frac{w}{T}\right), \tag{50}$$

by virtue of which, in the absence of heat sources (w = 0), J is a conservative quantity.

In isobaric coordinates, we can write the potential vorticity

$$J = \frac{1}{\rho_p} (\Omega_p + 2\sqrt{G}\omega) \nabla s, \tag{51}$$

where Ω_p is a vector with contravariant components

$$\Omega_p^\alpha = -\varepsilon^{\alpha\beta\gamma} [(\dot{z})_{,\beta} z_{,\gamma} + \tilde{u}_{\alpha,\beta}], \tag{52}$$

$\varepsilon^{\alpha\beta\gamma}$ - are the Levy-Civita symbols, and $\{\tilde{u}_\gamma\} = \{\dot{x}, \dot{y}, 0\} = \{u^1, u^2, 0\}$.

The negative sign in Eq. (52) takes account of the sinistral orientation of the bases $\{\mathbf{e}_\alpha\}$ and $\{\mathbf{e}^\alpha\}$. The earth's angular velocity vector has the covariant components

$$\omega^1 = 0,\ \omega^2 = \omega_y,\ \omega^3 = -\frac{1}{\sqrt{G}}(\omega_z - \omega_y z_{,y}). \tag{53}$$

The balance of vorticity in isobaric coordinates is

$$\dot{J} = \frac{1}{\rho_p} \frac{\partial}{\partial x^\alpha} \left[(\Omega_p^\alpha + 2\sqrt{G}\omega^\alpha) \frac{w}{T}\right]. \tag{54}$$

As will be seen from formulas (51) and (52), the potential vorticity in isobaric coordinates is extremely complex. Note that when the angular velocity ω is replaced in the equations of motion by its vertical component $\omega_z \mathbf{i}_z$, the same change takes place in the formula for the potential vorticity. For quasistatic equilibrium, the formula for the potential vorticity is considerably simplified:

$$J = \frac{1}{g} [u^2_{,3} s_{,1} - u^1_{,3} s_{,2} - (u^2_{,1} - u^1_{,2} + 2\omega_z) s_{,3}]$$

(here, in addition to quasistatic conditions, we also assume that $\omega = \mathbf{i}_z \omega_z$).

In transforming the equations for the dynamics of the atmosphere we neglected internal friction and other nonpotential forces. Addition of such forces as a supplementary term f in the definition (24) introduces no fundamental complications. It entails the incorporation into Eqs. (29a)-(29c) of the force components $\mathbf{f}\nabla z$, $\mathbf{e}^1\mathbf{f}$, $\mathbf{e}^2\mathbf{f}$, while $\rho_p \mathbf{v}\mathbf{f}$ (v appears in the right side of energy balance equation (47) (v is the absolute velocity (10), (14)), and a term $\frac{1}{\rho_p}\frac{\partial}{\partial x^\alpha}(s\varepsilon^{\alpha\beta\gamma} f_{\gamma,\beta})$, must be added to the right side of Eq. (54), where f_γ are the covariant components of f in isobaric coordinates. But it should be noted that the internal structure of the components of the dissipative forces is likely to be very complex in isobaric curvilinear coordinates.

Received December 7, 1988

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