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BOUNDARY VALUE PROBLEM FOR NONHYDROSTATIC, PRESSURE–COORDINATE GEOPOTENTIAL HEIGHT EQUATION

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SUMMARY

In a nonhydrostatic, anelastic, p–space model the geopotential height (GPH) field is determined by the Poisson equation. Physically relevant boundary conditions are formulated for this equation. Kinematical conditions for velocity at the ground and lateral boundaries generate nonhomogeneous Neumann condition for the GPH. At the ground the nonhomogeneous term includes, beside internal forces, terms, caused by the surface curvature. The finiteness of the solution is required at $p = 0$ in the case of finite sources of the Poisson equation. In an isothermal atmosphere this generates nonhomogeneous Dirichlet condition at the ground (which is supplementary to the Neumann condition). In the vertically nonhomogeneous atmosphere an isothermal thin shell is separated at upper boundary and the solution finiteness is required in the shell, which yields the nonhomogeneous radiative condition for the GSP at the upper boundary of the atmosphere below the shell. For sources finite at $p = 0$, the depth of the shell can shrink to zero, for nonrestricted source this is impossible and the analytical continuation of the solution to the top shell is required. This is the case where the GPH distribution becomes essentially nonhydrostatic at $p \to 0$ for spatial scales, commonly treated as belonging to the hydrostatic domain. The application of the radiative condition at the top requires modification of the ground pressure treatment. Two particular ground pressure models are discussed. One treats the ground pressure as completely adjusted to the GPH distribution. In the second case the short–wave component of the ground pressure is adjusted to the nonhydrostatic component of the GPH, while the long–wave component evolves like in the ordinary, hydrostatic, primitive–equation model.

1. Introduction

The p–space was introduced by Eliassen (1949), its adaptation to the non-uniform ground conditions was developed by Phillips (1957) in the σ–coordinate form. The representation of hydrostatic (HS) dynamics in pressure-coordinates became instantly popular and is dominating in large-scale atmospheric dynamics up to date, especially in climate modeling and weather forecast. Recently, the growing resolution of both numerical forecast and climate models as well as the growing requirements to the model precision has brought the transition from hydrostatic p–space dynamics to nonhydrostatic (NH) modeling into limelight.

The first formulation of nonhydrostatic (NH) dynamics in pressure–coordinates is given by Miller (1974) and Miller and Pearce (1974). The Miller–Pearce model abandons the hydrostatic equilibrium assumption in favour of the full vertical momentum equation but postulates the incompressibility of motion in pressure–space and in this way filters the acoustics. In this respect the Miller-Pearce model represents a version of anelastic models. The Miller–Pearce model requires a separation of statical background (sounding) and dynamical fluctuative temperature components. As demonstrated by Rõõm (1997a,b), this causes an additional restriction to the model, as the energy conservation requires the constant Brunt–Väisälä frequency of the background state. A generalization, free of restriction of such kind, is presented by White (1989). The White model employs the actual temperature to the full extent. A further modification of the NH p–space dynamics is designed by Salmon and Smith (1994). The Salmon–Smith model makes use of the Hamiltonian variational formulation of dynamics and supports the optional thermodynamics of the atmosphere. The Miller-Pearce model was originally designed in p–coordinates, σ–coordinate versions were developed by Miller and White (1984), and used in numerical modeling by Xue and Thorpe (1991), Miranda and James (1992), and Rõõm (1997a). Another, the so–called elastic filtered model, is proposed by Rõõm and Ülejõe (1996), and generalized by Rõõm (1997b). This model supports the compressibility in p–space and uses approximation of infinitely high sound speed for acoustic relaxation. In all referred models the actual pressure in an air particle is treated as the vertical coordinate of the particle. Still, there exist different approaches employing the hydrostatic component of pressure field (Laprise, 1992) or the hydrostatic mean background pressure field in the role of a vertical coordinate (Dudhia, 1993).

A common quality of all filtered, nonhydrostatic, pressure–space models is that they include a diagnostical field – the nonhydrostatic geopotential or geopotential height (GPH) – which is an analogue of the pressure field in the common coordinates. The GPH distribution is governed by a generalized Poisson equation. A central problem of the solution of this equation is the choice of boundary conditions. Commonly the homogeneous Neumann boundary condition is applied at all boundaries (see an overview by Miranda, 1990). A different approach is developed and tested in the sigma–coordinate version of the Miller–Pearce model by Rõõm (1997a). Ground surface and lateral boundaries are treated as ideal constraints generating nonhomogeneous Neumann condition for the GPH. At the top the condition of the regularity of solution (finiteness for a finite source function) is applied in the Fourier space which results in a nonhomogeneous radiative boundary condition for the GPH. In the present study, we reexamine the problem in detail in the pressure–coordinate version. The use of p–coordinates makes the treatment of the problem more transparent. Particularly, the treatment of the ground pressure and its role in the nonhydrostatic dynamics gains in clarity.

2. Anelastic, nonhydrostatic p–space model

2.1. Initial equations we employ are developed by Salmon and Smith (1994) in pressure coordinates $\{x, y, p\}$, and represented by Rõõm (1998) in the form

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\,g\nabla z \;-\; f\hat{\mathbf{z}}\times\mathbf{v}\;, \tag{1a}$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} = g\left(1 + \frac{p}{H}\frac{\partial z}{\partial p}\right)\;, \tag{1b}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = Q/T \;\equiv\; A_s\;, \tag{1c}$$

$$\nabla\cdot\mathbf{v} + \frac{\partial\omega}{\partial p} = 0\;, \tag{1d}$$

$$w = -\,H\left(\frac{\omega}{p} + \frac{A_s}{R}\right)\;. \tag{1e}$$

Here the vector $\mathbf{v}$ is horizontal velocity and w represents vertical velocity; z is the height of the isobaric surface (the GPH); g is the gravitational acceleration; f is the Coriolis parameter; $\hat{\mathbf{z}}$ is the vertical unit vector. H defines the height scale:

$$H = \frac{p}{g\rho} \qquad \left(= \frac{RT}{g} \right) ,$$

where ρ is the material density in the ordinary Cartesian space The first definition here is general while the definition in brackets corresponds to perfect gas; T is temperature and R is gas constant. s represents entropy and Q is the heat source. $\omega = dp/dt$ is vertical velocity in pressure coordinates.

It is assumed at the deduction of the model (1) that the enthalpy density $F(p,s)$ is the known function of p and s, and, in accordance with the general definition,

$$\frac{\partial F}{\partial p} = \frac{1}{\rho} = \frac{gH}{p}, \qquad \frac{\partial F}{\partial s} = T \, .$$

In the special case of perfect gas

$$F(p,s) = c_p T = c_p T_n \left(\frac{p}{p_n} \right)^{R/c_p} \mathrm{e}^{s/c_p} \, .$$

with T_n and p_n being constants (normal temperature and pressure), c_p is isobaric specific heat.

The represented Salmon–Smith model is different from the original model by White on which it is based. The Salmon–Smith model does not require perfect gas. In addition, the vertical velocity is approximately defined by formula $w = -(1/g)\mathrm{d}F/\mathrm{d}t$, which yields the final diagnostical formula (1e). Such approximation allows to maintain the energy conservation law in the model at variational formulation (Salmon and Smith 1994, Rõõm 1998). Meanwhile, in the context of the present study these differences are not too important. Particularly, both models coincide for perfect gas in the absence of heat sources.

(1a) – (1c) are prognostic equations for velocity and entropy, (1d) is the diagnostical equation for z, (1e) represents the relationship for ω determination, if w is known from (1b). The knowledge of ω is required for material derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \omega \frac{\partial}{\partial p} \, .$$

Such an interpretation – common velocity w is the prime field and ω–velocity is computed diagnostically afterwards – is needed for the development of the explicit diagnostical equation for z. In the final version w can be eliminated and ω computed from diagnostical equation (1d).

2.2. Boundary conditions. In the case of exact, nonfiltered dynamics (Rõõm 1990, 1997b) the domain in the p–space is

$$\mathbf{x} \in \Omega , \quad 0 < p < p_0(\mathbf{x},\ t) , \tag{2a}$$

where Ω is a two–dimensional domain (the complete xy–plain, or part of it, or the surface of the sphere), and $p_0(\mathbf{x},\ t)$ represents the ground surface pressure (time–dependent orography in the p–space). The surface pressure evolves in accordance with equation (expressing condition that the lower boundary surface is material in the p–space and consists of the same material particles all the time)

$$\frac{\mathrm{d}p_0}{\mathrm{d}t} = \omega|_{p_0} . \tag{2b}$$

Boundary conditions at surface are

$$z|_{p_0(\mathbf{x},t)} = h(\mathbf{x}) , \tag{2c}$$

where h is the ground surface height in the common Cartesian space, and the following from (2c) kinematical constraint for velocity (the slipping condition at the ground)

$$w|_{p_0(\mathbf{x},\ t)} = \mathbf{v}|_{p_0(\mathbf{x},\ t)} \cdot \nabla h(\mathbf{x}) . \tag{2d}$$

These exact boundary conditions will be applied at the deduction of the ground boundary condition for z. Still, lower boundary conditions will be discussed and revised further, in section 8.

3. Geopotential height equation

Diagnostical equation (1d) is an implicit equation for z (for other dependent fields there exist explicit prognostic or diagnostical equations). To get an explicit relationship, (1d) must be differentiated by t and local time derivatives of velocity components should be replaced from (1a), (1b)(*). The resulting equation reads

$$\nabla^2 z \ + \ \frac{\partial}{\partial p}\left[\frac{p}{H}\left(1 \ + \ \frac{p}{H}\frac{\partial z}{\partial p}\right)\right] \ = \ A_z \ , \tag{3}$$

where the source function is

$$A_z \ = \ -\frac{1}{g}\frac{\partial}{\partial p}\left[p\left(w\frac{\partial}{\partial t}\frac{1}{H} \ + \ \frac{1}{R}\frac{\partial A_s}{\partial t} \ - \ \frac{\hat{a}w}{H}\right)\right] \ - \ \frac{1}{g}\nabla\cdot(\hat{a}\mathbf{v} \ + \ f\hat{\mathbf{z}}\times\mathbf{v}) \ ,$$

and

$$\hat{a} \ = \ \mathbf{v}\cdot\nabla \ + \ \omega\frac{\partial}{\partial p}$$

The explicit expression for local time derivative of $1/H$ in the definition of A_z is

$$\frac{\partial}{\partial t}\frac{1}{H} \ = \ -\frac{1}{H^2}\,\frac{p}{g}\frac{\partial}{\partial p}\frac{\partial h}{\partial t} \ = \ -\frac{1}{H^2}\,\frac{p}{g}\frac{\partial}{\partial p}\left[T\left(\hat{a}s \ + \ A_s\right)\right] \ .$$

Relation (3) represents the diagnostical equation for the complete geopotential height. When z is expanded to the HS and NH components:

$$z \ = \ z_s \ + \ z_n \ , \tag{4}$$

where z_s is the common HS component of the geopotential height:

$$\frac{p}{H}\frac{\partial z_s}{\partial p} \ + \ 1 \ = \ 0 \ , \tag{5}$$

then (3) transforms to the equation for the nonhydrostatic geopotential height z_n:

$$\nabla^2 z_n \ + \ \frac{\partial}{\partial p}\left(\frac{p^2}{H^2}\frac{\partial z_n}{\partial p}\right) \ = \ A_z \ - \ \nabla^2 z_s \ \equiv \ A_n \ . \tag{6}$$

(*) In mechanical terms (see Serrin 1959), (1d) represents an ideal constraint upon the system and z is the Lagrangian multiplier, or the potential of the reaction force, which is required to keep the motion on the surface (1d) in functional space for successive instants, if the system initially belongs to this surface. The procedure of deduction of the diagnostical equation for the Lagrangian multiplier (z in our case, p in ordinary anelastic models) is general for all anelastic models.

Equations (3) and (6) are equivalent, still, equation (6) is more preferable. The HS component is not a solution of equation (3), but of the different equation (5). Differently from (3), elliptical operator on the left side of equation (6) is negatively definite. That means it has non–constant solutions, only if either the right hand source A_n is different from zero or the boundary conditions for z_n are nonhomogeneous (at least on a part of the boundary). This quality easily permits to estimate the domain of relevance of z_n for processes with different spatial scales. And, of course, it brings out the boundary value problem for z_n.

4. The reduced final model with the eliminated vertical velocity w

Now, after the z–equation (3) or (6) is deduced, it is possible to eliminate the common velocity w along with depending on it equations (1b) and (1e) (these two equations were required just for the deduction of equation (3)). The reduced model is very close in appearance to the common HS primitive equation model. It consists of equations (1a) for $\mathbf{v}$, (1c) for s, (1d) for ω, (5) for z_s and (4) and (6) for z and z_n. Essentially new in this model in comparison with the common HS equations is that the common HS height z_s is replaced by z and the system includes an additional equation (6) for nonhydrostatic height correction z_n. Still, despite apparent similarity, the described model is nonhydrostatic to full extent, and vertical velocity is implicitly persistent. For instance, the definition of energy (Rõõm 1998) includes kinetic energy of vertical motion, $w^2/2$.

5. Boundary condition for z_n at the ground

Like equation (1d), kinematical condition at the ground, (1d), represents an ideal constraint. This constraint states that material particles which are initially at the lower boundary surface will remain there for successive moments. To achieve this, the reaction of the surface is added to the ordinary forces at the surface. The reaction force of the surface creates the normal gradient of z_n. As a consequence, additional accelerations arrive at the surface, and are continued via z–equation to the internal points of the domain, forcing the real movement keep to boundary constraints.

To get a quantitative relationship, we apply the material "surface" derivative

$$\frac{\mathrm{d}_0}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v}|_{p_0} \cdot \nabla$$

to (2d):

$$\frac{\mathrm{d}_0 w|_{p_0}}{\mathrm{d}t} = \nabla h \cdot \frac{\mathrm{d}_0 \mathbf{v}|_{p_0}}{\mathrm{d}t} + (v^\alpha v^\beta)_{p_0} h_{\alpha\beta} \ . \tag{7}$$

The short notation is used here

$$(v^\alpha v^\beta)_{p_0} h_{\alpha\beta} \equiv \sum_{i,j=1}^{2} (v^i v^j)_{p_0} \cdot \frac{\partial^2 h}{\partial x^i \partial x^j} \ .$$

If p_0 evolves in accordance with (2b), then

$$\frac{\mathrm{d}_0}{\mathrm{d}t} a|_{p_0} = \left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)_{p_0}$$

for optional smooth a. Using this formula along with identity

$$(\nabla z)_{p_0} = \nabla h - \left(\frac{\partial z}{\partial p}\right)_{p_0} \nabla p_0 \ ,$$

and equations of motion (1a) and (1b), condition (7) can be transformed to the condition

$$\left[\left(\frac{p}{H} - \nabla h \cdot \nabla p_0\right) \frac{\partial z_n}{\partial p}\right]_{p_0} =$$

$$\underset{A}{-\,(f/g)\nabla h \cdot (\hat{\mathbf{z}} \times \mathbf{v})_{p_0}} \;\; \underset{B}{+\,(v^\alpha v^\beta)_{p_0} h_{\alpha\beta}/g} \;\; \underset{C}{-\,\nabla h \cdot \left(\frac{H}{p}\nabla p_0 + \nabla h\right)_{p_0}} \equiv \gamma \ . \tag{8}$$

The derived relationship represents the nonhomogeneous Neumann boundary condition for z_n. On the right side are the forces which the vertical gradient of z_n should compensate to avoid intersections of particle trajectories with the ground. Nonhomogeneity is to the full extent caused by orography and disappears in case of flat ground.

Term (A) in (8) is caused by the Coriolis force.

Term (B) describes the influence of acceleration, caused by the curvature of the ground. $h_{\alpha\beta}$ is the curvature tensor of orography(*). As this curvature–term is second order in

(*) For a spherical, flat (not steep) hill with the orography $h(x,y) = \sqrt{(r+h_0)^2 - x^2 - y^2} - r \approx h_0 - (x^2+y^2)/(2r), \quad h_0, x, y << r$, where h_0 is the height of the hill and $r + h_0 \approx r$ is the radius of curvature, the curvature tensor is $h_{\alpha\beta} = -\delta_{\alpha\beta}/r$.

$\mathbf{v}$, it becomes large not only for steep orography but increases rapidly with the rise of wind speed, too.

The last term (C) is determined by the baroclinity of the atmosphere at the ground. This term disappears (for nonuniform ground) if p_0 and h are related by the barometric formula with barotropic (independent of horizontal coordinates) height scale $H(p)$:

$$h = \int_{p_0}^{a} \frac{H(p)}{p} dp .$$

As the solution of equation (6) becomes constant (being a zero for physical reasons) for homogeneous boundary conditions, assuming that sources A_n are absent, it has maximum values near the surface (again, if A_n is zero) for nonhomogeneous conditions. Thus, (8) can be used for estimation of the amplitude of z_n due to orographic effects.

6. Lateral boundary conditions for z_n

If the model is horizontally limited it has lateral boundary Σ. Usually the horizonal velocity field is predetermined at this boundary. For z_n, the distribution of the normal component is essential

$$v_n|_\Sigma \equiv \mathbf{n} \cdot \mathbf{v}|_\Sigma = a(\xi, t) , \quad \xi \in \Sigma \tag{9}$$

where $\mathbf{n}$ is the unit vector, normal to Σ, as the normal component determines the flow through boundary, which causes additional stresses in the internal points of the domain. Usually the flow is assumed to be in balance:

$$\int_\Sigma \mathbf{v} \cdot \mathbf{n} d\sigma = \int_\Sigma a(\xi) d\sigma = 0 .$$

Kinematical condition (9) represents an ideal constraint, similar to (2d). Still, differently from the previous case, the lateral boundary is not material, which means that individual air particles are not restricted to the surface Σ, but leave (penetrate) it. Due to this special feature, it is more straightforward to apply for (9) the local time derivative rather than the material derivative at the deduction of the boundary condition for z.

After the local derivative of $\mathbf{v}$ is substituted from the equations of motion, the resulting lateral boundary condition for normal derivative of z_n reads

$$g\left(\frac{\partial z}{\partial n}\right)_\Sigma = -\mathbf{n}\cdot\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + f\hat{\mathbf{z}}\times\mathbf{v}\right)_\Sigma =$$

$$-\left[\frac{\partial a}{\partial t} + a\left(\frac{\partial v_n}{\partial n}\right)_\Sigma + v_s|_\Sigma\frac{\partial a}{\partial s} + \omega|_\Sigma\frac{\partial a}{\partial p} - fv_s|_\Sigma\right] . \qquad (10)$$

Here v_s is the component of horizontal velocity, tangent to the boundary Σ. This is the boundary condition for the complete geopotential height. For the NH height fluctuation the boundary condition reads

$$g\left(\frac{\partial z_n}{\partial n}\right)_\Sigma = -\mathbf{n}\cdot\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + f\hat{\mathbf{z}}\times\mathbf{v} + \frac{\partial z_s}{\partial n}\right)_\Sigma .$$

Note that the second and the third term at the right side are mutually compensated for the geostrophic flow.

The most important contribution to the nonhomogeneous right side is given by the second term in the brackets in (10). This term becomes large if the normal component of velocity has a large normal gradient, which can easily happen at numerical modeling, if special care is not taken to dump large normal gradients at lateral boundaries.

7. Second "horizontal" boundary condition for z_n

We do not recommend to use condition (2c) at the bottom as the second "horizontal" boundary condition supplementing the "first", (8), as this would yield solutions with "explosive" growth at $p \to 0$. Rather we deduce the wanted condition from the principle that the growth of z_n at zero is of the same order than the growth of source A_n in (6), and particularly, that z_n is finite at zero for a restricted source.

To get quantitative relations, is suitable to transform equation (6) along with the first boundary condition (8) into the Fourier space in horizontal coordinates $\mathbf{x}$ and study the remaining boundary condition separately for each Fourier component. This lays for finite domain Ω an additional restriction that this domain should be rectangular, for nonlimited plain or spherical geometry there are no such restrictions. Another shortcoming with Fourier transform is that the transformed equation has simple structure (separation of different mode–equations) for horizontally homogeneous conditions with independent of $\mathbf{x}$ coefficients (H in our case) and boundary condition (8). To overcome this problem we restrict the consideration to cases where the operator on the left side in (6) and boundary condition (8) can be divided to the main horizontally homogeneous parts and small corrections to them, which can be included iteratively. For this we represent (6) and (8) in the equivalent forms

$$\nabla^2 z_n \;+\; \frac{\partial}{\partial p}\left(\frac{p^2}{\overline{H}^2}\frac{\partial z_n}{\partial p}\right) \;=\; A_n \;+\; \mathcal{M} z_n \;\equiv\; A^* \;, \tag{11a}$$

$$\mathcal{M} z_n \;=\; -\frac{\partial}{\partial p}\left[p^2\left(\frac{1}{H^2} \;-\; \frac{1}{\overline{H}^2}\right)\frac{\partial z_n}{\partial p}\right] \tag{11b}$$

and

$$\left(\frac{p}{\overline{\overline{H}}}\frac{\partial z_n}{\partial p}\right)_{\overline{p_0}} \;=\; \gamma \;+\; \mathcal{N} z_n \;\equiv \gamma^* \;, \tag{12a}$$

$$\mathcal{N} z_n \;=\; -\;\left[\left(\frac{p}{H} \;-\; \nabla h\cdot\nabla p_0\right)\frac{\partial z_n}{\partial p}\right]_{p_0} \;+\; \left[\frac{p}{\overline{\overline{H}}}\frac{\partial z_n}{\partial p}\right]_{\overline{p_0}} \tag{12b}$$

here $\overline{p_0}$ represents the mean value of p_0 over domain Ω. For real atmospheric conditions, the terms $\mathcal{M} z_n$ and $\mathcal{N} z_n$ are small corrections to the main parts on the left sides of (11a) and (12a), which can be easily included iteratively. The procedure is studied in detail by Miranda (1990) and Rõõm (1997a). For the boundary problem, an extensive study of the iterative problem is not needed and the detailed structure of perturbation operators $\mathcal{M}$, $\mathcal{N}$ is not required. All we need is the assumption that the perturbations are small enough to maintain the upper boundary condition of the nonperturbed (horizontally homogeneous) problem. In this case problem (11a), (12a) can be studied for optional right–hand terms A_n^* and γ^* with the help of the Fourier transform in $\mathbf{x}$. Transformed equation and lower boundary condition are

$$\frac{\partial}{\partial p}\left(\frac{p^2}{\overline{H}^2}\frac{\partial z_k}{\partial p}\right) \;-\; k^2 z_k \;=\; A_k^* \;, \tag{13a}$$

$$\left(\frac{p}{\overline{\overline{H}}}\frac{\partial z_k}{\partial p}\right)_{\overline{p_0}} \;=\; \gamma_k^* \;, \tag{13b}$$

where z_k, A_k^* and γ_k^* are Fourier coefficients of z_n, A^* and γ^* consequently, and $k^2 = k_x^2 + k_y^2$ represents the square of the modulus k of the wave vector $\mathbf{k}$.

7.1. The case of constant height scale. At first we will study the second boundary condition problem for the constant $\overline{H}$, the case of p–dependent $\overline{H}$ will be studied after that. Equation (13a) can be easily integrated for the constant $\overline{H}$.

For $k = 0$ the general solution is

$$z_0(p) \;=\; z_0(\overline{p_0}) + \overline{p_0}\left(\frac{\partial z_0}{\partial p}\right)_{\overline{p_0}} + \overline{H}^2\int_p^{\overline{p_0}} A_0^* \frac{dp'}{p'} - \frac{\overline{p_0}}{p}\left[\overline{p_0}\left(\frac{\partial z_0}{\partial p}\right)_{\overline{p_0}} + \frac{\overline{H}^2}{\overline{p_0}}\int_p^{\overline{p_0}} A_0^* dp'\right] \tag{14}$$

If $k \neq 0$, then

$$z_k(p) \;=\; z_k^1(p) \;+\; z_k^2(p)\;, \tag{15a}$$

$$z_k^i(p) \;=\; \left(\frac{\overline{p_0}}{p}\right)^{\mu_i}\left[(a_i \;-\; b_i)\, z_k(\overline{p_0}) \;-\; b_i\overline{p_0}\left(\frac{\partial z_k}{\partial p}\right)_{\overline{p_0}} + \; b_i\overline{H}^2\int_p^{\overline{p_0}}\left(\frac{p'}{\overline{p_0}}\right)^{\mu_i} A_k^*(p')\frac{dp'}{p'}\right]\;, \tag{15b}$$

$$a_1 \;=\; \frac{\mu_1}{\mu_1 - \mu_2}\;,\quad a_2 \;=\; \frac{\mu_2}{\mu_2 - \mu_1}\;, b_1 \;=\; -\; b_2 \;=\; \frac{1}{\mu_1 - \mu_2}\;,$$

$$\mu_1 \;=\; 1/2 \;-\sqrt{1/4 \;+\; \overline{H}^2\mathbf{k}^2}\;,\quad \mu_2 \;=\; 1/2 \;+\sqrt{1/4 \;+\; \overline{H}^2\mathbf{k}^2}\;.$$

Note. If $A_k^* \neq 0$, then z_k^1 are z_k^2 not solutions separately, they perform independent solutions only for the homogeneous equation (13a) without source. The grouping of the solution in the form (15a) for non–zero source A_k^* takes into account the different asymptotic behaviour of functions z_k^1 and z_k^2.

For $k = 0$, the general solution includes term proportional to $1/p$ (the last term in (14)), to which the exponential growth with height corresponds in common Cartesian coordinates. The specific quality of that component is that it occurs in the general solution for the homogeneous case, $A* \equiv 0$, too. Such exponential growth is nonphysical. Corresponding term we will call the irregular component of the general solution in contradiction with the remaining regular solution. The main task is to get physically relevant regular solutions by exclusion (elimination) of irregular ones. In the present case with $k = 0$, the regularization is achieved by equalizing with zero $A_0^*(p)$ and γ_0^*. As $A_0^*(p)$ and γ_0^* represent the horizontally averaged components of $A^*(\mathbf{x}, p)$ and $\gamma(\mathbf{x})$, the horizontally averaged sources and nonhomogenuity of boundary condition should be absent. Such a restriction is physical but it represents a constraint to the model which is beyond the boundary value problem; regularization of $z_0(p)$ cannot be achieved with the choice of integration constants. In this respect the case $k = 0$ is exceptional. Anyway, for $A_0^*(p) = 0$, $\gamma_0^* = 0$, solution (14) decays to constant $z_0(\overline{p_0})$, which should put zero on the physical consideration$^{(*)}$.

The situation is different for $k \neq 0$. Here the irregularity is associated with the function z_k^2, which has the growth tendency (amplitude) $\sim (\overline{p_0}/p)^{\mu_2} > \overline{p_0}/p$. For the regularization it is necessary to put the coefficient of $(\overline{p_0}/p)^{\mu_2}$ zero at $p = 0$:

$$(a_2 - b_2)\, z_k(\overline{p_0}) - b_2 \overline{p_0} \left(\frac{\partial z_k}{\partial p} \right)_{\overline{p_0}} + b_2 \overline{H}^2 \int_0^{\overline{p_0}} \left(\frac{p'}{\overline{p_0}} \right)^{\mu_2} A_k^*(p') \frac{dp'}{p'} = 0 \,. \qquad (16)$$

Using the definitions of a_2 and b_2, this condition can be presented as

$$\overline{p_0} \left(\frac{\partial z_k}{\partial p} \right)_{\overline{p_0}} - (\mu_2 - 1)\, z_k(\overline{p_0}) = - \overline{H}^2 \int_0^{\overline{p_0}} \left(\frac{p'}{\overline{p_0}} \right)^{\mu_2} A_k^*(p') \frac{dp'}{p'} \,. \qquad (16')$$

As gradient $\partial z_k/\partial p$ at the ground is determined by (13b), 16') represents the condition for $z_k(\overline{p_0})$. Thus, for $k \neq 0$ and constant $\overline{H}$, both "horizontal" boundary conditions are given at the ground surface.

If condition (16) is satisfied, the regularized solution can be presented in the form

$$z_k(p) = C \left(\frac{p}{\overline{p_0}} \right)^{-\mu_1} + b_1 \overline{H}^2 \left[I_k^1(p) + I_k^2(p) \right] \,, \qquad (17a)$$

where

$$C = (a_1 - b_1)\, z_k(\overline{p_0}) - b_1 \overline{p_0} \left(\frac{\partial z_k}{\partial p} \right)_{\overline{p_0}} \qquad (17b)$$

$$I_k^1(p) = \int_p^{\overline{p_0}} \left(\frac{p'}{p} \right)^{\mu_1} A_k^*(p') \frac{dp'}{p'} \,, \quad I_k^2(p) = \int_0^p \left(\frac{p'}{p} \right)^{\mu_2} A_k^*(p') \frac{dp'}{p'} \,. \qquad (17c)$$

The homogeneous part of this solution, determined by C (i.e., by the boundary conditions) and corresponding to absence of sources A_k^*, is always decreasing with the height, as $-\mu_1 > 0$. The regularity of the nonhomogeneous part of the regularized solution

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(*) Mode $k = 0$ represents a disturbance with the infinite wavelength and belongs to the hydrostatic domain. The proved zero–solution of z_0 supports the simple physical fact that NH disturbances are absent at the long–wave hydrostatic limit.

(17) near the upper boundary $p = 0$ is entirely determined with the regularity of source $A_k^*(p)$. Let a test source have exponential growth with the height:

$$A_k^*(p) \;=\; \left(\frac{p}{\overline{p_0}}\right)^{q \,-\, \mu_2} \quad . \tag{18}$$

If $q \leq 0$, then integral I_k^2 (and consequently the solution) does not exist. For $q \;>\; 0$ solution exists and

$$I_k^1(p) \;=\; \begin{cases} \frac{1}{q \,+\, \mu_1 - \mu_2}\left[\left(\frac{p}{\overline{p_0}}\right)^{-\mu_1} \;-\; \left(\frac{p}{\overline{p_0}}\right)^{q-\mu_2}\right] & q \;\neq\; \mu_2 - \mu_1 \\ \left(\frac{p}{\overline{p_0}}\right)^{-\mu_1} \ln\frac{\overline{p_0}}{p} & q \;=\; \mu_2 - \mu_1 \end{cases}$$

$$I_k^2(p) \;=\; \frac{1}{q}\left(\frac{p}{\overline{p_0}}\right)^{q-\mu_2} \quad .$$

As we see, even the regularized solution can grow exponentially with the height, if the source is increasing. This happens for $0 < q < \mu_2$. For $q > \mu_2$ the solution decreases with height. The growth rate of the regularized solution coincides with the growth rate of the source.

7.2. Height–dependent $\overline{H}$. The radiative boundary condition. For $k \,=\, 0$ there are no significant changes in comparison with the previous case, except that $\overline{H}^2$ should be placed in (14) in the integrals. The main conclusion of the homogeneous $\overline{H}$ is maintained: to get a regular solution, A_0^* and q_0^* should be zero.

Essential modification is necessary for $k \neq 0$. We shall use a special "top shell" method. Suppose $\overline{H}$ is a restricted smooth function near p =0. Then $\overline{H}(p)$ may be treated as the constant $\overline{H}(0)$ in some vicinity of the upper boundary $0 \;<\; p \;<\; p_t$ (in the top shell). In this vertically homogeneous top shell the previous solution is applicable, if $\overline{p_0}$ is everywhere replaced by p_t. For the regularity of solution condition (16') should hold at the lower boundary of the top shell (independently of the behaviour of the solution below p_t)

$$p_t\left(\frac{\partial z_k}{\partial p}\right)_{p_t} \;-\; (\mu_2 \;-\; 1)\, z_k(p_t) \;=\; -\,\overline{H}^2(0)\frac{1}{p_t^{\mu_2}}\int_0^{p_t} (p')^{(\mu_2-1)}\, A_k^*(p')dp' \;. \tag{19}$$

At the same time, (19) is the upper boundary condition for the atmosphere below level p_t. In fact, this is the nonhomogeneous generalization of the widely used "radiative" boundary condition. Thus, in the lower, nonhomogeneous atmosphere the problem can be solved numerically with "horizontal" boundary conditions (13b) and (19), and continued analytically into the top shell.

For source A_k^*, finite at zero, it is possible to consider the limiting case of disappearing top shell, modifying (19) as

$$\lim_{p \to 0} \left[p \frac{\partial z_k}{\partial p} - (\mu_2 - 1)\, z_k \right] = - \frac{\overline{H}^2(0) A_k^*(0)}{\mu_2} \,. \tag{19'}$$

Still, this is not possible for source, unlimited at zero. For instance, in the case of a test source (analogical to (18))

$$A_k^*(p) = \left(\frac{p}{p_*} \right)^{q - \mu_2}$$

with the (independent of p_t) constant p_*, the right side of (19) becomes infinite at $p_t \to 0$, if $0 < q < \mu_2$, though the regularized solution exists. If source function has such a "bad" growth, either the top shell of finite depth must be employed or the source should be modified upon physical assumptions.

The systematical growth of the source with height is observable (realistic) in many cases. An example is presented by free orographic waves, for which $A_k^* \sim 1/p^{1/2}$. In such cases the amplitude of the z_k is increasing with the height, and there exists a region in the upper atmosphere where nonhydrostatic correction z_n becomes significant and comparable with fluctuations of the hydrostatic height distribution even at scales, which are traditionally treated as the hydrostatic ones.

Boundary condition (19) has been employed and tested in a numerical model (Rõõm 1997a). The testing results are positive.

Evidently, boundary condition (19) is not restricted to nonhomogeneous stratification but can be employed equally for the atmosphere with constant $\overline{H}$. In this case (19) and (16') present equivalent boundary conditions.

8. Revision of the lower boundary treatment

The developed radiative upper boundary condition (19) and the initial boundary condition (2c) are contradictory and eliminate each other. The initial boundary condition (2c), which in the context of the present investigation was necessary for the foundation of the kinematical condition (2d), represents in reality the boundary condition for the exact, acoustically nonfiltered dynamics. In the present anelastic, acoustically filtered model this boundary condition leads to nonphysical amplitudes with irregular growth of the nonhydrostatic geopotential height fluctuation. This is obvious from the solution (15). In turn, the use of regularizing boundary conditions (16') or (19) removes boundary condition (2c).

We support the point of view (which is still hypothetical and requires future detailed study of acoustical adjustment process with emphasis on the ground pressure relaxation) that irregular modes, if such tend to develop, are governed by acoustical processes and are removed from the system in the course of acoustical relaxation. As a result, the ground surface pressure is adjusted to the regular height distribution of the z_n. That means, the condition (2c) is an equation for the determination of the ground pressure p_0 from the known geopotential height distribution $z(\mathbf{x},\ p,\ t)$. Consequently, evolution equation (2b) is disregarded, which (in agreement with the ground pressure adjustment assumption) assumes the elimination of long transient ground pressure (i.e., mass) waves from the model.

The presented ground pressure treatment assumes complete acoustic relaxation. If one wants to maintain the evolutional development of long nonbalanced mass waves, the following approximative model is straightforward. We will use the circumstance that the NH component z_n and the barotropic component of the HS height z_s are spectrally well–separated. The spatial scale of z_n is less than 30 km (usually even less than 10 km), while the typical scale of the barotropic component of z_s (which is responsible for ground pressure waves) is about 1000 km or larger. This permits to associate the acoustical adjustment of the ground pressure with its short–scale component. For that we present the ground pressure as the sum of hydrostatic long–wave component p_s and small NH short–wave correction p_n to it:

$$p_0(\mathbf{x},\ t)\ =\ p_s(\mathbf{x},\ t)\ +\ p_n(\mathbf{x},\ t)\ . \tag{20}$$

The geometry of the domain is governed by the long–wave component (this approximation is correct because both the mean and time–variable components of p_s are much larger by the amplitude than p_n):

$$\mathbf{x}\ \in\ \Omega\ ,\qquad 0\ <\ p\ <\ p_s(\mathbf{x},\ t)\ , \tag{21a}$$

$$\frac{\mathrm{d}p_s}{\mathrm{d}t}\ =\ \omega|_{p_s}\ , \tag{21b}$$

i.e. instead of the full ground pressure in (2a) – (2c), here stands the hydrostatic surface pressure. The last evolution equation can be presented with the help of (1d) as the vertically integrated mass conservation law

$$\frac{\partial p_s}{\partial t}\ +\ \nabla\cdot\int_0^{p_s}\mathbf{v}dp\ =\ 0\ . \tag{21b'}$$

The small nonhydrostatic ground pressure correction p_n in (20), supplementary to the main hydrostatic component, is estimated with the help of (2c) via z_n. Extrapolation of the left hand term of (2c) from level p_s yields

$$z_s[\mathbf{x},p_s(\mathbf{x},\ t),t]\ +\ z_n[\mathbf{x},p_s(\mathbf{x},\ t),t]\ +\ p_n(\mathbf{x},\ t)\left(\frac{\partial z(\mathbf{x},\ p,\ t)}{\partial p}\right)_{p_s(\mathbf{x},\ t)}\ =\ h(\mathbf{x})\ .$$

As

$$z_s[\mathbf{x},p_s(\mathbf{x},\ t),t]\ =\ h(\mathbf{x})\ ,$$

and approximately

$$\left(\frac{\partial z}{\partial p}\right)_{p_s}\ \approx\ \left(\frac{\partial z_s}{\partial p}\right)_{p_s}\ =\ -\ \left(\frac{H(\mathbf{x},\ p,\ t)}{p}\right)_{p_s}\ ,$$

we get for p_n

$$p_n(\mathbf{x},\ t)\ =\ p_s(\mathbf{x},\ t)\ \frac{z_n(\mathbf{x},p_s(\mathbf{x},\ t),t)}{H(\mathbf{x},p_s(\mathbf{x},\ t),t)}\ . \tag{22}$$

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