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**ACOUSTIC FILTERING
IN NONHYDROSTATIC PRESSURE COORDINATE
DYNAMICS: A VARIATIONAL APPROACH**

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ABSTRACT

Nonhydrostatic, acoustically filtered model of atmospheric dynamics in pressure coordinates is derived using a special filtering technique. The initial complete nonhydrostatic equations in pressure space are linearized. The linearized system is divided into two subsystems – to independent equation for potential vorticity, which determines the quasigeostrophic horizontal flow and is a local invariant in the absence of heat sources, and the fourth order wave system describing acoustical and buoyancy waves. A Lagrangian function, corresponding to the wave equations, is derived. The acoustic filtering is carried out in the Lagrangian. The approximated Lagrangian generates filtered wave equations and the linear filtered equations of motion. As a final step, the linear model is extended to a nonlinear nonhydrostatic acoustically filtered model by inclusion of advection terms in vorticity equation and by nonlinear generalization of the Hamiltonian principle for the wave system. Thus, variational principles are employed for both the acoustical filtration and nonlinear extension of the filtered approximation, which guarantees the maintenance of conservational qualities of initial model in the final filtered version. The deduced dynamical model has no previous analogues.

1. Introduction

A continuous medium that lacks acoustic disturbances can be characterised as an acoustically adjusted or acoustically relaxed one. In slow atmospheric processes the acoustic modes are not significantly activated, and the atmosphere is usually acoustically relaxed. Meanwhile, acoustic modes are potentially present (they can be activated) and, as a consequence, the complete hydrodynamic equations, which are capable of describing motions in an extremely broad domain of spatial and temporal scales, support beside slow motions also fast acoustic solutions.

As is well-known, the potential existence of acoustic waves and perturbations among solutions of hydrodynamic equations is a source of problems in numerical integration. Because acoustic disturbances propagate in the atmosphere with the sound speed (≈ 350

m/s), their explicit modeling requires small time steps in accordance with the Courant–Friedrichs–Lewy stability requirement. The time–steps in explicit algorithms must be two or three orders of magnitude smaller than it would be necessary for the adequate resolution of the slow processes.

There are two general methods in use for dealing with this acoustic instability problem. Following the terminology of Tanguay, Robert and Laprise (1990) they may be called the numerical relaxation and the physical filtering methods.

The numerical relaxation technique is the younger one; see, for example, Tapp and White 1976, Klemp and Wilhelmson 1978, Cotton and Tripoli 1978, Durran and Klemp 1983. Klemp and Wilhelmson 1978 use the time-splitting method for the acoustic component: in equations responsible for acoustic evolution a shorter time step is used than in equations describing the slow development. This allows relaxation of acoustic modes to a quasi-equilibrium state within every larger time-step. Tapp and White 1976 make use of a semi-implicit scheme to control acoustic oscillations. Terms involving propagation of sound waves are treated implicitly, while other terms are represented explicitly. The method assumes that the initial state of the atmosphere is acoustically relaxed (if there is no special interest in the investigation of acoustic modes, of course). It has been developed further in an investigation by Tanguay, Robert and Laprise 1990, who apply the implicit acoustic–component integration scheme of Tapp and White in combination with the semi-Lagrangian algorithm. This results in a conditionally stable integration scheme with large time steps. Though phase speeds of acoustic modes are distorted for time steps which do not satisfy Courant–Friedrichs–Lewy stability requirement, that has little influence on the slow dynamics, if the initial conditions are acoustically relaxed.

The other method, the physical filtering method, is the older one: The hydrodynamic equations are simplified or approximated in such a way that they do not include acoustic solutions anymore. A severe restriction, which makes physical filtering a nontrivial task, is the requirement that the slow dynamics should be maintained undistorted or almost undistorted in the filtered model. Classical representatives of this family are the anelastic equations (Ogura and Charney 1962) and the hydrostatic primitive equations (HPE) in pressure coordinates (Eliassen 1949).

The family of different models based on the anelastic approximation is quite numerous and includes besides the classical incompressible Navier–Stokes model and Boussinesq’ equations the shallow– and deep–convection models (Ogura and Phillips 1962), which employ the ”shallow” and ”deep” continuity equations (Dutton and Fichtl 1969, Pielke 1984). The most recent addition to the anelastic family is the ”super-deep convection model” (our terminology) of Durran 1989, in which the anelastic continuity equation is replaced by a more general pseudo-incompressible condition of thermodynamical origin.

The development of anelastic models has proceeded from small to larger spatial scales i.e. from models designed for small-scale processes (incompressible Navier-Stokes, shallow convection models) to models which are capable of describing beside small processes also medium (deep convection model) and large (super-deep convection model) scale events. An analogous development but in the opposite direction – from large to smaller scales – has gone on in the p -space model family. The main motivation for such development has been the wish to generalize initially hydrostatic pressure-space models so that they will describe nonhydrostatic effects, without abandoning p -coordinates. In contrast to their hydrostatic counterparts, the existing nonhydrostatic (NH) pressure–coordinate models are not always acoustically filtered, so it is essential to distinguish acoustically filtered and complete NH models in p -space.

The first and probably best known NH model in pressure space is the Miller–Pearce model, hereafter the MPM (Miller 1974, Miller and Pearce 1974). This model abandons the hydrostatic relationship in favour of the nonhydrostatic vertical momentum equation, but it postulates the incompressibility of motion in pressure–space and in this way it filters the acoustic modes. A generalization of the MPM is presented by White 1989. In the White model (WM) the horizontally homogeneous background temperature of the original MPM is replaced by the actual temperature field. The MPM was originally designed in p -coordinates. Sigma–coordinate versions were developed by Miller and White 1984 and used in numerical modeling by Xue and Thorpe 1991 and Miranda and James 1992. The most general representation for nonhydrostatic nonfiltered hydrodynamic equations in pressure coordinates is presented by Rõõm 1989, 1990. This model, hereafter referred to as the RR, may be deduced with the help of the direct transformation of complete nonfiltered equations, using the curvilinear coordinate co-

variant differencing formalism, from ordinary space to the p -space. It does not assume any preliminary simplification (including the preservation of the full Coriolis force). Invariant definitions of energy density and potential (Ertel) vorticity and the consequent conservation laws are presented in Røed 1990. As will be demonstrated later in this paper, the WM and the MPM are filtered approximations to these equations.

All the p -coordinate models so far cited, beginning with the Eliassen model and finishing with the RR, employ for their vertical coordinate the actual pressure. However, there exist models which employ different pressure coordinates. The first one, developed by Laprise (1992), employs the hydrostatic pressure rather than the full thermodynamic pressure. Another such model is the nonhydrostatic extension of the Penn State – NCAR model (Dudhia 1993), in which the mean background pressure is the vertical coordinate, while the pressure fluctuation is treated as a dependent dynamic variable, which is a function of p -coordinates along with other dependent fields.

In the present paper our aim is to deduce acoustically filtered nonhydrostatic equations from the general RR equations. As was pointed above, filtered approximations to the RR are the the MPM and the WM, but – as we shall show – deduction of filtered equations from the RR in linear case reveals that these models are not the only possible models. As it turns out, the general initial model yields a diagnostic relationship between the velocity components, which can not hold simultaneously with the incompressibility condition used in the WM and MPM. Thus, there exist different models and the problem is, which of them to prefer and in which circumstances. This may be called an "optimal filtering" problem, and a key question is which should be the criteria for optimality. The answer depends to some degree on the tasks for which the filtered model is designed, but there exist some general points which are common for all models. First, the filtered model should maintain main conservation laws; second, it should not distort the slow dynamics, including buoyancy waves. The first criterion can be accepted without any restriction. At the same time the second criterion can be satisfied only approximately, because the filtered model represents an approximation and it has a residual error (in comparison with the exact model), which becomes larger the faster and more energetic is the modelled process. Therefore it is reasonable to specify the second criterion as a requirement of maximum accuracy for processes with

small amplitudes (or, in other words, for linear processes).

Summarising, the primary criteria for optimum filtering are:

- the filtered model must possess the same conservation characteristics as does its non-filtered counterpart;
- the filtered model must guarantee maximum accuracy for linear processes.

In such a specification the problem can be managed with the help of the least action principle and Hamiltonian principle. The development of a filtering technique which uses the variational principles of mathematical physics is the main topic of the present study.

Application of variational principles in the mechanics of continuous media has a long lasting history. There exist two different treatments of the Least action and Hamiltonian principles – the mechanical one and the field-theoretic one. The mechanical approach treats the continuous medium as a system of material particles, to which all mechanical principles can be applied. This trend begins with the investigations by Clebsch 1857, 1859, and it is essentially developed by Herivel 1955 and Lin (unpublished paper). Progress to the end of the 1950s is summarised in the fundamental monograph by Serrin 1959 (the unpublished paper of Lin is cited in this monograph). Important results obtained thereafter in atmospheric dynamics include the development of the Hamiltonian formalism for the shallow water model by Salmon 1983, for the HPE by Shutts 1989, and for the quasi-hydrostatic equations by Roulstone and Brice 1994, the deduction of the MPM from the least action principle by Salmon and Smith 1994, an application of the Poisson brackets formalism by Salmon 1988.

The field-theoretic interpretation of the variational principles is more common in other branches of theoretical physics and especially in linear field theory (see Landau and Lifchitz 1987). In this method the field is treated at every point as an independent oscillator and its magnitude is varied rather than the trajectories of material particles. It is common in this treatment to deduce the Lagrangian function from known field equations to get conservative characteristics of the field with the help of the Noether theorem.

In this paper we use variational methods which are more close to the field-theoretic

view-point. Variational technique is employed in respect to the subsystem which is responsible for wave processes in the continuous medium. Such a selective use of variational principles is the main explanation why the field-theoretic approach does the work. Both the least action principle which employs the Lagrangian formalism, and the Hamiltonian principle along with Hamiltonian formalism are used. The least action principle is used to get filtered linear equations, while the Hamiltonian principle is employed for nonlinear extension of the obtained model.

In short, the filtering of acoustic modes with the help of the least action principle is carried out as follows. The nonlinear RR are linearized, divided into a quasisolenoidal independently developing component and to the fourth order wave-system. The Lagrangian function is constructed for the wave-component. The filtering approximation is introduced immediately into the Lagrangian. Due to the simple structure of the Lagrangian, possible approximations that yield wave-filtration are not numerous and they are easily recognizable as they are associated with time-order lowering. In turn, the filtered Lagrangian generates the filtered linear dynamics, if one moves from the Lagrangian back to the equations with the help of the least action principle. The use of this principle guarantees in accordance with the Noether theorem maintenance of the conservation laws of the initial linearized model (assuming, of course, that the filtered Lagrangian has the same temporal and spatial symmetry as the original, nonfiltered Lagrangian).

As a final step, the linear filtered equations are extended to nonlinear forms. For that the Hamiltonian variational formulation of wave equations is used. The Hamiltonian formulation is more preferable in this case because it operates with the first order equations in time, which are closer to the initial nonfiltered system by appearance. The nonlinear extension consists in essence of the introduction of the density into Hamiltonian variational integral and complementing of local tendencies in the potential vorticity equation and in the variational integral.

Though the variational approach is general and its applicability does not depend on the choice of initial model or geometrical representation, in this investigation the filtering technique is designed keeping in mind the nonhydrostatic equations in the p -space. The general method is developed and realized on this specific example. As an applicable

output, two filtered sets of equations are proposed which are closely related and differ only in the treatment of nonlinear advection. In one scheme the medium is compressible in pressure coordinates (but still lacks acoustic disturbances), and the omega-velocity is determined in (approximate) accordance with its thermodynamical nature. In the second scheme the slow flow of the medium is approximated as incompressible and the omega-velocity is evaluated via the continuity equation of the incompressible medium. The relative accuracy of the two models and their domains of practical application will not be studied in this paper and serve as subjects for future investigation.

2. The complete set of NH equations in pressure coordinates

2.1. Height of the isobaric surface. The pressure field $p(x, y, z, t)$ consists of a hydrostatic main component, p_s , and an NH correction, p_n : $p = p_s + p_n$. Correspondingly, in pressure coordinates (\mathbf{x}, p) , $\mathbf{x} = (x, y)$, the height of an isobaric surface $z(\mathbf{x}, p, t)$ presents in a similar way (Fig. 1):

$$z(\mathbf{x}, p, t) = z_s(\mathbf{x}, p, t) + z_n(\mathbf{x}, p, t) , \quad (1a)$$

$$z_s(\mathbf{x}, p, t) = h(\mathbf{x}) + \frac{R}{g} \int_p^{p_0(\mathbf{x}, t)} \frac{T(\mathbf{x}, p', t)}{p'} dp' . \quad (1b)$$

Here $h(\mathbf{x}, p, t)$ is the height of the ground above sea-level, $p_0(\mathbf{x}, t)$ represents the atmospheric pressure field at the ground and $T(\mathbf{x}, p, t)$ is the temperature. The hydrostatic height component z_s corresponds to the height which the air-particle would have if the pressure of that particle were entirely determined by the hydrostatic effect, i.e., by the weight of the atmospheric column above the particle. The remaining part of the height, z_n , is defined as the difference of actual and hydrostatic heights of the particle. The independent height-coordinate, p , corresponds to the actual pressure.

The correction term z_n is entirely caused by the nonhydrostatic pressure deviation p_n . Because $|z_n| \ll z_s$ and $|p_n| \ll p$ in the atmosphere, the nonhydrostatic pressure and height corrections are related as (Fig. 2)

$$\frac{z_n}{H} \approx \frac{p_n}{p} \quad (2)$$

where $H = RT/g$ is a (variable) atmospheric height scale. For processes with infinitesimal amplitudes (which can be described in the framework of linearized models) this approximate equality may be replaced by the exact equality

$$\frac{z_n}{H} = \frac{p_n}{p} . \quad (2')$$

These formulae are useful for comparison of different pressure- and height-coordinate models, as they allow pressure gradient forces to be expressed as gradients of the geopotential height and vice versa.

2.2. General NH equations in p -coordinates. If the pressure field is a monotone function of height,

$$\frac{\partial p}{\partial z} < 0 ,$$

then it is possible to transform the dynamic equations of the atmosphere from Cartesian coordinates $\{x, y, z, t\}$ to pressure coordinates $\{x, y, p, t\}$, disregarding the hydrostatic assumption (see RR). The resulting complete, nonfiltered, nonhydrostatic p -coordinate equations can be presented after minor simplification of the Coriolis force in the form:

$$\frac{dz}{dt} = w \quad (3a)$$

$$n \frac{dw}{dt} = g(1 - n) , \quad (3b)$$

$$n \frac{d\mathbf{v}}{dt} = -g\nabla z - nf\mathbf{k} \times \mathbf{v} , \quad (3c)$$

$$\frac{dT}{dt} = \frac{RT\omega}{c_p p} + Q , \quad (3d)$$

$$\frac{dn}{dt} = -n(\nabla \cdot \mathbf{v} + \partial\omega/\partial p) , \quad (3e)$$

$$n = -\frac{p}{H} \frac{\partial z}{\partial p} . \quad (3f)$$

Here $\mathbf{v} = (u, v)$ and w are horizontal wind vector and vertical wind respectively, $\omega = dp/dt$ presents the omega-velocity of an air particle, n is a normalized, non-dimensional density in pressure coordinates, which is related to the ordinary air density as:

$$n\delta p = g\rho\delta z ,$$

where δp , δz represent vertical extents of a small (infinitesimal) air-particle in the pressure- and ordinary space; Q is the thermal forcing (heat source divided by c_p), \mathbf{k} represents the vertical unit vector, and the total (or Lagrangian) derivative is defined as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \omega \frac{\partial}{\partial p}$$

with time and horizontal derivatives taken at constant pressure.

2.3. Boundary conditions. Conditions at lateral boundaries are the same as in Cartesian coordinate models and so are not of special interest in the present study. The

main differences with the usual model occur in the "horizontal" conditions at the top and at the bottom. The domain occupied by the atmosphere is

$$0 < p < p_0(\mathbf{x}, t) \quad (4a)$$

where the lower boundary p_0 is not fixed, but evolves in accordance with the equation

$$\frac{dp_0}{dt} = \omega|_{p_0} , \quad (4b)$$

which expresses the condition that the lower boundary consists all the time of the same air particles¹⁾. Thus, the domain is varying in time and (4b) presents an additional evolutionary (prognostic) equation which must be integrated along with the system (3).

Boundary conditions at $p = p_0(\mathbf{x}, t)$ and $p = 0$ are

$$z|_{p_0} = h(\mathbf{x}) , \quad (5a)$$

$$\omega|_0 = 0 . \quad (5b)$$

The first assumes the existence of a rigid underlying surface in ordinary physical space; it yields (for slipping boundary) kinematical condition for vertical velocity at the surface

$$w|_{p_0} = \frac{dh}{dt} = \mathbf{v}_{p_0} \cdot \nabla h . \quad (5a')$$

The second defines a fixed boundary in the p -space at the level $p = 0$; this boundary condition forbids mass outflow to the cosmos in physical space.

2.4. Diagnostic equation for ω . Model (3) presents a system consisting of seven equations for seven fields z, u, v, w, T, n and ω . To close this system, the (two-dimensional) prognostic field p_0 must be included and the evolution equation (4b) added. All quantities of the model are prognostic fields except ω , and system (3), (4b) includes a single diagnostic equation (3f). This equation must be used for the determination of

¹⁾ If a continuous medium has a bounding surface, which moves in accordance with a differential equation governed by the state of that medium, this surface is called free. In this respect $p_0(\mathbf{x}, t)$ describes a free boundary and in the p -space the atmosphere is a continuous medium with free surface.

the diagnostic field ω ¹⁾. As (3f) does not include ω explicitly, the way to proceed is to differentiate (3f) by t and eliminate time derivatives by the help of other equations in system (3). The result is an explicit equation for ω

$$\alpha \frac{\omega}{p} - \frac{Q}{T} = \frac{p}{nH} \left(\frac{\partial w}{\partial p} - \frac{\partial \mathbf{v}}{\partial p} \cdot \nabla z \right) - \nabla \cdot \mathbf{v} \equiv -D, \quad (6)$$

where

$$\alpha = \frac{c_v}{c_p}.$$

In equation (6) the quantity on the right hand side, denoted as D , is the divergence of the three-dimensional velocity $\{u, v, w\}$ in common Cartesian space:

$$D = \left(\frac{\partial u}{\partial x} \right)_{y,z} + \left(\frac{\partial v}{\partial y} \right)_{x,z} + \left(\frac{\partial w}{\partial z} \right)_{x,y}.$$

Indeed, (6) is the general tendency equation for the thermodynamic pressure field, appearing in isobaric coordinate space as a diagnostic relationship. It is a standard form of the thermodynamic equation, written in terms of dp/dt upon application of the continuity equation and the perfect gas law. As the pressure plays the key role in acoustic adjustment, (6) will be an equation of primary significance for further treatment.

Now, after the diagnostic equation for ω has been derived, the initial relationship for its derivation, equation (3f), has played its role, and it may be dropped from further treatment. Still, there exist other possibilities – we can employ diagnostic equation (3f) for the elimination of one of the three dependent variables (z, n, T), and drop the evolution equation for that variable (ie. (3a), (3e) or (3d)) from consideration. This means we can go ahead with three different sets of equations, which differ in appearance but are all equivalent to each other, and are by one equation and one dependent variable less than the initial system. We shall use such a reduced model in section 3.4.

2.5. The hydrostatic primitive–equation limit. For motions with small vertical accelerations, $dw/dt \rightarrow 0$,

$$z \rightarrow z_s, \quad n \rightarrow 1, \quad (7)$$

¹⁾ This diagnostic equation represents a constraint that reduces the time-order of the model by one. Due to the (3f), only two of z, T, n can be initialized independently. Thus, the time-order of (3), (4b) is not seven (= number of time-derivatives) but six.

where z_s satisfies the hydrostatic condition (equivalent to definition (1b))

$$\frac{\partial z_s}{\partial p} = -\frac{H}{p}. \quad (8)$$

Equations (3) transform at this limit to the ordinary HPE model. Formally the HPE can be reached, substituting everywhere in (3) n by 1 and z by z_s . The continuity equation (3e) transforms at the HPE-limit to the condition of incompressibility,

$$\nabla \cdot \mathbf{v} + \partial\omega/\partial p = 0. \quad (9)$$

The hydrostatic analogue for (6) is

$$\alpha \frac{\omega}{p} - \frac{Q}{T} = \frac{p}{H} \left(\frac{\partial w_s}{\partial p} - \frac{\partial \mathbf{v}}{\partial p} \nabla \cdot z_s \right) - \nabla \cdot \mathbf{v}. \quad (10)$$

It can be deduced from the HPE in the same way which was used for the deduction of (6), if the definition

$$w_s = dz_s/dt \quad (11)$$

is assumed for the vertical velocity at hydrostatic limit. Note that (10) is very close in appearance to the original nonhydrostatic version (6) and can be deduced from (6), using limit (7). Though the HPE model does not need equation (10), this diagnostic relation may be used for the determination of the hydrostatic vertical velocity, defined by (11).

Due to the assumption of the incompressibility (9) the HPE model filters acoustic waves. An exception is presented by the external, or surface waves, which are supported when the pressure at the lower boundary evolves according to nonbalanced equation (4b).

2.6. The White and Miller–Pearce models. Other models which filter sound waves are the WM and MPM. The WM was alternatively derived from the Hamilton principle by Salmon and Smith 1994. Here we will outline the deduction of the WM from the general NH model (3).

The WM rests on two fundamental approximations, which are introduced into the initial model (3). The first one is the incompressibility approximation, $n = 1$, which is used everywhere, except the right side of equation (3b). Due to this approximation, equation (3e) transforms to the continuity relation for incompressible fluid, (9) (which filters

vertically propagating acoustic waves). The other approximation is the representation of the total derivative for z in (3a) as

$$\frac{dz}{dt} \approx \omega \frac{\partial z_s}{\partial p} = -H \frac{\omega}{p} .$$

This reduces the initial evolution equation (3) to the diagnostic relationship

$$\tilde{w} = -H \frac{\omega}{p} , \quad (12a)$$

where \tilde{w} stands for the approximate value of vertical velocity, which obviously differs (for given $\omega = dp/dt$) from both the exact definition (3a), w , and the quasi-static definition (11), w_s . In the momentum equations the density n is approximated by the unit value except the right side of (3b), where it is expressed using (3f):

$$\frac{d\tilde{w}}{dt} = g \left(1 + \frac{p}{H} \frac{\partial z}{\partial p} \right) , \quad (12b)$$

$$\frac{d\mathbf{v}}{dt} = -g \nabla z - f \mathbf{k} \times \mathbf{v} . \quad (12c)$$

Finally, the thermodynamic equation maintains its initial appearance

$$\frac{dT}{dt} = \frac{RT\omega}{c_p p} + Q . \quad (12d)$$

Equations (9), (12) represent the WM (White 1989).

In contrast to the HPE model, which has an adjusted analogue (10) for equation (6), the WM lacks such an analogue. This occurs due to the use of the (more restrictive) approximation (12a) instead of the "natural" assumption (11)¹⁾. This difference is the main motivation to look further, in section 3, for acoustically filtered models which, differently from the WM, preserve the diagnostic relationship (6).

The MPM follows from the WM after presentation $T = T_0(p) + T'$, $z = z_0(p) + z'$ and linearization of (12) in temperature and geopotential height fluctuations:

$$\tilde{w} = -H_0 \frac{\omega}{p} , \quad (13a)$$

¹⁾ The difference in (11) and (12a) exhibits also that the WM is generally not physically identical to the HPE at the long-wave limit.

$$\frac{d\tilde{w}}{dt} = g \left(\frac{T'}{T_0} + \frac{p}{H_0} \frac{\partial z'}{\partial p} \right), \quad (13b)$$

$$\frac{d\mathbf{v}}{dt} = -g\nabla z' - f\mathbf{k} \times \mathbf{v}. \quad (13c)$$

$$\frac{dT'}{dt} = \frac{T_i\omega}{p} + Q = -T_0 \frac{N^2}{g} \tilde{\omega} + Q. \quad (13d)$$

Here

$$H_0 = RT_0/g, \quad T_i = \frac{R}{c_p} T_0 - p \frac{\partial T_0}{\partial p}, \quad N = \sqrt{RT_i}/H_0 \quad (14)$$

represent the height scale, the stability parameter, and the Väisälä frequency of the background state.

The original MPM (Miller and Pearce 1974) employs, differently from the presented version (9), (13), the mean potential temperature Θ_0 and its fluctuation Θ' rather than T_0 , T' . Still, due to the relationship $\Theta'/\Theta_0 = T'/T_0$, differences in these two versions are not essential.

Additional approximation, which is characteristic to both the WM and MPM, and has been overlooked in former investigations, concerns the lower boundary dynamics. Due to the diagnostic relationship (12a) or (13a), equation (4b) is replaced by

$$\text{WM :} \quad \frac{dp_0}{dt} = -p_0 \left(\frac{\tilde{w}}{H} \right)_{p_0}, \quad (15a)$$

$$\text{MPM :} \quad \frac{dp_0}{dt} = -p_0 \left(\frac{\tilde{w}}{H_0} \right)_{p_0}, \quad (15b)$$

These equations exhibit that in the WM and MPM the lower boundary dynamics in the p -space is mainly dependent on the vertical velocity on the ground and thus, it is determined by mechanical rather than by thermodynamical processes. Particularly, in the MPM it follows from (15b) that the lower boundary is fixed in p -space and coincides with the barometric mean value of surface pressure \bar{p}_0 :

$$\bar{p}_0(\mathbf{x}) = a \exp \left[-\frac{g}{R} \int_0^{h(\mathbf{x})} \frac{dz}{T_0(z)} \right] \quad (15c)$$

supposing the evolution equation for \tilde{w} is (5a'). Any departure of p_0 from \bar{p}_0 at initial instant yields controversy with (13a) at the lower boundary. Thus, the MPM filters surface pressure waves completely. In the WM some weak lower boundary evolution is

possible due to the departure of H at right side of (15a) from the mean H_0 . Still, it is a good idea to choose p_0 at initialization equal to its mean value (15c). The complete elimination of lower boundary motion in p -space by the MPM and reduction of its amplitude by the WM does not mean the elimination of dynamic fluctuations of surface pressure. As it follows from (2), ground pressure fluctuations can be evaluated via geopotential height fluctuations at the mean surface pressure level \bar{p}_0 : $p'_0 = \bar{p}_0 z'|_{p_0}/H_0(p_0)$.

2.7. Energy conservation. The energy conservation law can be presented for (3) in the form

$$n \frac{d}{dt} e + \nabla \cdot (\mathbf{v}gz) + \frac{\partial}{\partial p}(\omega gz) = nc_p Q$$

where the energy density is

$$e = \frac{\mathbf{v}^2}{2} + \frac{w^2}{2} + c_p T + \left(1 - \frac{1}{n}\right)gz .$$

The last term here represents energy which the atmosphere has due to its (unrestricted) compressibility. In both the HPE and the WM this term is absent. In the WM the energy density takes the form

$$e = \frac{\mathbf{v}^2}{2} + \frac{\tilde{w}^2}{2} + c_p T ,$$

and the HPE lacks the vertical kinetic energy too:

$$e = \frac{\mathbf{v}^2}{2} + c_p T .$$

In the MPM the energy density takes a form, more close to the wave energy definitions of linear models:

$$e = \frac{\mathbf{v}^2}{2} + \frac{\tilde{w}^2}{2} + \frac{1}{2} \left(\frac{gT'}{NT_0} \right)^2 .$$

It must be pointed out that energy conservation in the MPM is not a general characteristic of the model, but (as first demonstrated by Rõõm, 1997) it is restricted to model situations in which term $N(p)T_0(p)$ is a conservative characteristic of material particle, i.e.

$$\frac{d}{dt}(NT_0) = 0$$

(if the potential temperature is employed instead of T , like in the original MPM (Miller and Pearce 1974), then $N(p)\Theta_0(p)$ should be conservative). Three model situations

when this condition holds, are: a) Np_0 is independent (beside the common independence of t and \mathbf{x}) of vertical coordinate p ; b) model is fully linearized: $d/dt \rightarrow \partial/\partial t$; c) Lagrangian time derivative includes the mean, background horizontal advection only: $d/dt \rightarrow \partial/\partial t + U(p)\partial/\partial x$.

3. Linear model

3.1. Linearization of equations (3) according to the hydrostatic equilibrium state, characterised by the mean temperature, $T_0(p)$, yields equations

$$\frac{\partial z'}{\partial t} = w + H_0 \frac{\omega}{p}, \quad (16a)$$

$$\frac{\partial w}{\partial t} = -g n', \quad (16b)$$

$$\frac{\partial \mathbf{v}}{\partial t} = -g \nabla z' - f \mathbf{k} \times \mathbf{v}, \quad (16c)$$

$$\frac{\partial T'}{\partial t} = \frac{T_i \omega}{p} + Q, \quad (16d)$$

$$\frac{\partial n'}{\partial t} = -(\nabla \cdot \mathbf{v} + \partial \omega / \partial p), \quad (16e)$$

$$n' = -\left(\frac{p}{H_0} \frac{\partial z'}{\partial p} + \frac{T'}{T_0} \right). \quad (16f)$$

Here z' and T' and n' represent isobaric height, temperature and density fluctuations, H_0 and T_i are defined by (14).

The domain occupied by the atmosphere in the p -space is fixed in linear case:

$$0 < p < \bar{p}_0(\mathbf{x}), \quad -\infty < x, y < \infty,$$

where \bar{p}_0 is defined by (15c). Boundary conditions at the bottom and top are

$$w|_{\bar{p}_0} = \frac{dh}{dt} = \mathbf{v}|_{\bar{p}_0} \cdot \nabla h, \quad \omega|_0 = 0.$$

The first one represents an extrapolation to the mean lower boundary \bar{p}_0 of the exact relation $w|_{p_0} = dh/dt$, which follows from (3a) and (5a), the second coincides with (5b).

Pressure fluctuations on the ground, p'_0 , can be evaluated via height fluctuations (see (2')):

$$p'_0 = \bar{p}_0 \left(\frac{z'}{H_0} \right)_{\bar{p}_0} . \quad (17a)$$

There exists another relationship for the pressure fluctuation, which represents the linearized version of (4b) and which may be presented with the help of (15c) as

$$\frac{\partial p'_0}{\partial t} = \left(\omega + \frac{pw}{H_0} \right)_{\bar{p}_0} . \quad (17b)$$

Still, (17b) can be deduced from (17a) with the help of (16a) and thus, it does not represent an independent evolution equation.

3.2. Separation of the linear model to the wave system and potential vorticity equation.

Diagnostic equation for ω can be deduced in the same way as for the nonlinear case and the resulting expression coincides with the linearized version of the equation (6):

$$\alpha \frac{\omega}{p} = \frac{p}{H_0} \frac{\partial w}{\partial p} - \nabla \cdot \mathbf{v} + \frac{Q}{T_0} . \quad (18)$$

This equation along with (16f) enable us to get from (16) a reduced set of equations, fifth order in time, and divide this reduced model into two subsystems. One subsystem represents horizontal flow governed by the independent potential vorticity equation. Another is a fourth order subsystem, which describes wave processes (the wave system).

We introduce nondimensional fields ζ and η in place of z' and T' :

$$\zeta = \frac{z'}{H_0} , \quad \eta = \frac{T'}{T_0} - \frac{T_i z'}{T_0 H_0} . \quad (19)$$

ζ presents a relative height fluctuation, scaled in H_0 , η can be identified as the relative fluctuation of the entropy. Namely, let us define the nondimensional entropy as the function of the nondimensional potential temperature, Θ :

$$S(\Theta) = \ln \Theta , \quad \Theta = \frac{T}{T_a} \left(\frac{a}{p} \right)^{R/c_p} ,$$

where T_a and a are constants (the mean temperature and pressure at the sea-level), and let $S_0(z)$ present the background hydrostatic value of S as a function of the geometric height:

$$S_0(z) \equiv S[\Theta_0(z)] .$$

We define the relative entropy as a difference in its actual and background values at the same pressure level:

$$s(\mathbf{x}, p, t) = S[\Theta(\mathbf{x}, p, t)] - S_0[z(\mathbf{x}, p, t)] \equiv \varphi[\Theta(\mathbf{x}, p, t), z(\mathbf{x}, p, t)] . \quad (20)$$

Defined in this way, s is a known functional $\varphi(\Theta, z)$ of the potential temperature and height of the particle. This relative entropy is zero for background conditions:

$$s_0(p) \equiv s|_{\Theta_0(p), z_0(p)} = \varphi[\Theta_0(p), z_0(p)] = 0 ,$$

and for small temperature and height perturbations it coincides with the field η defined by (19):

$$\begin{aligned} s' &= \left(\frac{\partial \varphi}{\partial \Theta} \right)_{\Theta_0, z_0} \Theta' + \left(\frac{\partial \varphi}{\partial z} \right)_{\Theta_0, z_0} z' = \\ &= \frac{\Theta'}{\Theta_0} + \frac{dS_0}{dz_0} z' = \frac{T'}{T_0} - \frac{T_i}{T_0 H_0} z' , \end{aligned}$$

because $dS_0/dz_0 = T_i/(T_0 H_0)$.

Thus, η represents in linearized model, like s in the nonlinear case, the difference between actual entropy of an air particle and the value which the air particle would have at the same height in the background atmosphere.

Using the new field variables, equations (16a), (16b) and (16d) can be presented as

$$\alpha \frac{\partial \zeta}{\partial t} = \frac{1}{H_0} \hat{P}^+ w - b + \frac{Q}{T_0} , \quad (21a)$$

$$\frac{\partial \eta}{\partial t} = - \frac{N^2 w}{g} + \frac{Q}{T_0} , \quad (21b)$$

$$\frac{\partial w}{\partial t} = g \left(\hat{P}^- \zeta + \eta \right) . \quad (21c)$$

where

$$b \equiv \nabla \cdot \mathbf{v}$$

and short notations are introduced for operators

$$\hat{P}^+ = p \frac{\partial}{\partial p} + \alpha , \quad \hat{P}^- = \frac{\partial}{\partial p} p - \alpha .$$

Supplemented with equation (16c), (21a) – (21c) give a closed reduced system for η , ζ , w , and \mathbf{v} . This represents a straightforward way for Lagrangian representation of the

linear dynamics, if the Coriolis force is absent, $f = 0$; and this is the way which was used in Rõõm and Ülejõe, 1996 (see Rõõm 1997 as well). Still, in the presence of the Coriolis force the problem is a little bit more sophisticated, and the proceeding requires a transformation of (16c) to two scalar equations for the horizontal wind divergence

$$\frac{\partial b}{\partial t} = -gH_0\nabla^2\zeta + f \left[-q + \alpha f\zeta + f \left(\frac{\partial}{\partial p} p \frac{T_0}{T_i} - 1 \right) \eta \right] , \quad (21d)$$

and for the potential vorticity q :

$$\frac{\partial q}{\partial t} = f \frac{\partial}{\partial p} \left(\frac{p}{T_i} Q \right) , \quad (22a)$$

where

$$q = -\mathbf{k} \cdot (\nabla \times \mathbf{v}) + \alpha f\zeta + \frac{fg}{H_0} \hat{P}^- \frac{\eta}{N^2} . \quad (22b)$$

Equations (21d) and (22a) follow from (16c), (21a) – (21c) at the additional assumption $f = \text{const}$. The defined potential vorticity q represents the linearized version of the Ertel potential vorticity (which is presented for complete p -space dynamics in RR). It is an essential quality of the system that q evolves independently of other fields in accordance with (22a) and is uniquely determined by the heat sources Q . For an adiabatic process it does not change in time at all and represents a locally invariant field. Consequently, in wave processes participate only remaining four fields, governed by wave system (21). The wave system is tuned by the potential vorticity, which acts in equation (21d) as external forcing. In short-scale region, where f becomes effectively zero, the tuning of the wave system by the potential vorticity disappears and two subsystems become fully independent (in linear approximation).

If equations (21) and (22) are solved, horizontal velocity can be find via the stream function ψ and potential φ using wellknown representation

$$\mathbf{v} = \mathbf{k} \times \nabla\psi + \nabla\varphi . \quad (23a)$$

At that for ψ and φ equations follow from the definitions of b and q

$$\nabla^2\psi = -q + \alpha f\zeta + \frac{fg}{H_0} \hat{P}^- \frac{\eta}{N^2} , \quad (23b)$$

$$\nabla^2\varphi = \hat{P}^+ w + \frac{Q}{T_0} . \quad (23c)$$

3.3. Second order wave equations. It is easy to get two second order equations for ζ and η , by differentiating (21a) and (21b) w.r.t. time and eliminating the first order time derivatives with the help of (21c) and (21d):

$$\left\{ H_0^2 \left[\nabla^2 - \left(\frac{1}{c_a} \frac{\partial}{\partial t} \right)^2 - \frac{f^2}{c_a^2} \right] + \hat{P}^+ \hat{P}^- \right\} \zeta + \hat{P}^+ \left(1 - \frac{f^2}{N^2} \right) \eta = -Q_\zeta, \quad (24a)$$

$$\left[\left(\frac{1}{N} \frac{\partial}{\partial t} \right)^2 + 1 \right] \eta + \hat{P}^- \zeta = Q_\eta, \quad (24b)$$

where

$$Q_\zeta = \frac{R}{g^2} \frac{\partial Q}{\partial t} + \frac{H_0 f}{g} q, \quad Q_\eta = \frac{1}{N^2} \frac{\partial Q}{\partial t} \frac{Q}{T_0},$$

and $c_a = \sqrt{RT_0/\alpha}$ is the sound speed¹⁾. Equations (24) are wave equations for η and ζ . These equations can be employed for the modeling of linear wave processes in p -coordinate representation in a general, nonfiltered case.

Because parameter f^2/N^2 in (24a) is very small, the approximation $1 - f^2/N^2 \approx 1$ looks natural. Though this approximation is reasonable at scales < 100 km, it is not a good idea to apply it in the synoptic domain, as this may cause serious distortion of orographic-wave spectrum at scales > 1000 km.

3.4. The Lagrangian function and energy. The significance of second order equations (24) for the present study is that they have a Lagrangian function \mathcal{L} and can be deduced with the help of the least action principle

$$\delta S = \delta \int_{t_0}^{t_1} dt \int_{\Omega} dx dy dp \mathcal{L} = 0,$$

¹⁾ Following the tradition of wave-equation representations we have chosen the sound speed c_a for the prime acoustic characteristic of the atmosphere and the Väisälä frequency N for the prime characteristic of the buoyancy. Though perhaps the most relevant in physical terms, such a choice is not the best from the point of view of the symmetry (which is always important in Lagrangian formalism). For the maximum symmetry either the characteristic frequency $N_a = c_a/H_0$ instead of c_a , or the characteristic buoyancy-wave phase-speed $c_i = \sqrt{RT_i} = NH_0$ instead of N , should be used.

as extremes of the Lagrangian action S . The Lagrangian \mathcal{L} is supposed to be a function of field variables ζ , η and their derivatives:

$$\mathcal{L} = \mathcal{L}(\zeta, \zeta_t, \zeta_x, \zeta_y, \zeta_p; \eta, \eta_t, \eta_x, \eta_y, \eta_p),$$

where $\zeta_t = \partial\zeta/\partial t$, $\zeta_x = \partial\zeta/\partial x$, etc. are short notations for partial derivatives. Action S is varied in variations $\delta\zeta(\mathbf{x}, p, t)$, $\delta\eta(\mathbf{x}, p, t)$, which vanish at the boundaries of the domain Ω and at the initial and final moments, t_0 and t_1 . The condition of extremity $\delta S = 0$ for arbitrary $\delta\zeta$ and $\delta\eta$ yields Lagrangian equations

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\zeta} - \frac{\partial}{\partial t} \frac{\partial\mathcal{L}}{\partial\zeta_t} - \frac{\partial}{\partial x} \frac{\partial\mathcal{L}}{\partial\zeta_x} - \frac{\partial}{\partial y} \frac{\partial\mathcal{L}}{\partial\zeta_y} - \frac{\partial}{\partial p} \frac{\partial\mathcal{L}}{\partial\zeta_p} &= 0, \\ \frac{\partial\mathcal{L}}{\partial\eta} - \frac{\partial}{\partial t} \frac{\partial\mathcal{L}}{\partial\eta_t} - \frac{\partial}{\partial x} \frac{\partial\mathcal{L}}{\partial\eta_x} - \frac{\partial}{\partial y} \frac{\partial\mathcal{L}}{\partial\eta_y} - \frac{\partial}{\partial p} \frac{\partial\mathcal{L}}{\partial\eta_p} &= 0, \end{aligned}$$

which must coincide with the wave equations (24). To ensure this, it is sufficient to choose the Lagrangian function \mathcal{L} in the form

$$\mathcal{L} = \mathcal{T} - \mathcal{V}, \quad (25a)$$

where the generalized kinetic and potential energy densities, \mathcal{T} and \mathcal{V} , are

$$\mathcal{T} = \frac{1}{2} \left(\frac{H_0}{c_a} \zeta_t \right)^2 + \frac{1}{2} \left(1 - \frac{f^2}{N^2} \right) \left(\frac{\eta_t}{N} \right)^2, \quad (25b)$$

$$\begin{aligned} \mathcal{V} = \frac{1}{2} \left\{ \left(\frac{fH_0\zeta}{c_a} \right)^2 + (H_0\nabla\zeta)^2 + \left[\hat{P}^-\zeta + \left(1 - \frac{f^2}{N^2} \right) \eta \right]^2 + \right. \\ \left. \left(1 - \frac{f^2}{N^2} \right) \frac{f^2}{N^2} \eta^2 \right\} - \zeta Q_\zeta - \left(1 - \frac{f^2}{N^2} \right) \eta Q_\eta \end{aligned} \quad (25c)$$

The purpose of our Lagrangian formalism is to provide us with the necessary tools for optimal acoustic filtering. On the one hand, the existence of the Lagrangian guarantees energy conservation for the linearized system. On the other hand, with the help of the Lagrangian formalism it is easy to get filtered versions of the model which are still energy-conserving. System (24) possesses (according to the Noether theorem) wave energy

$$E_{\mathcal{L}} = \int_V \mathcal{H} \, dx dy dp,$$

with the density

$$\mathcal{H} = \zeta_t \frac{\partial \mathcal{L}}{\partial \zeta_t} + \eta_t \frac{\partial \mathcal{L}}{\partial \zeta_t} - \mathcal{L} = \mathcal{T} + \mathcal{V}. \quad (26)$$

$E_{\mathcal{L}}$ is conservative, if the system is isolated from external forcing, i.e., if $Q = 0$ and $q = 0$.

The least action principle uses the Lagrangian function, \mathcal{L} , as the prime field and results in two Lagrangian equations (24), which are both second order in time. Alternatively, there exists (see Salmon 1983, for example) another, Hamiltonian formulation of the problem, which in that formulation is called the Hamiltonian principle. The Hamiltonian formulation uses \mathcal{H} as the prime function and results in four first order equations, which are related to the fourth order wave system (24). In this paper the Hamiltonian equations are required at final extension of filtered model to the nonlinear case. For that reason they are presented for the resting background conditions in the Appendix A and generalized to the moving medium in the Appendix B.

4. Acoustic filtering

For slow atmospheric movements with small Mach number,

$$\mathcal{F} \equiv U^2/c_a^2 \ll 1,$$

where U is the characteristic amplitude of velocity, it is reasonable to filter the model acoustically, i.e. to simplify the equations so that they do not include acoustic-wave solutions anymore, though maintaining other waves and slow movements. Essentially the filtering consists of lowering the time order of the system by two. The filtering task can be solved in a most straightforward manner using the Lagrangian formalism. The main idea is that filtering (ie. time-order reduction) should be carried out in the Lagrangian function, which must be approximated so that the resulting wave equations do not include acoustic-wave solutions. As the approximate model has still the Lagrangian function, it supports the energy conservation law and other conservative qualities. The filtered wave equations with the conserving wave energy are the main output of the filtered Lagrangian function.

4.1. Filtered Lagrangian and wave equations. The most straightforward filtering approximation is $c_a \rightarrow \infty$ in the kinetic energy definition (25b). This model has been

introduced by Rõõm and Ülejõe (1996); here it will be presented in an extended version. Though most simple and transparent, this filtering approximation is still not the only possible filtering approximation in the Lagrangian. In (Rõõm 1997) an alternative case is studied in which the approximation of time derivatives ζ_t , η_t in (25b) with their hydrostatic values is used, and which results in the MPM.

If the Mach number is small, then the first term in \mathcal{T} is small in comparison with two first terms in \mathcal{V} in all spatial scales and, thus, in the first approximation it can be neglected. The resulting expression for the Lagrangian function is (25a) with the kinetic energy density

$$\mathcal{T} = \frac{1}{2} \left(1 - \frac{f^2}{N^2} \right) \left(\frac{\eta_t}{N} \right)^2 ,$$

and potential energy density (25c).

Filtered wave equations, corresponding to this Lagrangian, can be deduced directly from (24) with the help of the formal limit $c_a \rightarrow \infty$ in equation (24a):

$$\left\{ H_0^2 \left[\nabla^2 - \frac{f^2}{c_a^2} \right] + \hat{P}^+ \hat{P}^- \right\} \zeta = - \hat{P}^+ \left(1 - \frac{f^2}{N^2} \right) \eta - Q_\zeta , \quad (27a)$$

$$\left[\left(\frac{1}{N} \frac{\partial}{\partial t} \right)^2 + 1 \right] \eta + \hat{P}^- \zeta = Q_\eta . \quad (27b)$$

As a result, the time order of the system is reduced by two and two solutions of the four of the initial model are filtered. The solutions eliminated in this way are acoustic modes. At the same time, buoyancy waves are maintained, as the wave equation, "responsible" for their existence, (27b), maintains its initial appearance and still includes the Väisälä frequency N .

The main consequence of the filtering is that the wave equation (24a) is replaced by the Poisson equation in ζ , (27a). This equation defines ζ as a quasi-static field which is determined by the forcing on the right side of the equation. The name "quasi-static" is justified, because the solution ζ of equation (27a) coincides for stationary forcing with the exact static (time-independent) solution of the exact nonfiltered equations (24) in stationary conditions. For nonstationary conditions the quasi-static ζ depends on time parametrically, via Q and η (and, perhaps, via nonstationary boundary conditions).

In general, the quasi-static ζ is different from the hydrostatic height fluctuation ζ_s , which corresponds to the fluctuative part of the hydrostatic height, (1b), and which

represents a solution of the hydrostatic equation. The hydrostatic equation reads in terms of η , ζ (instead of T' , z'):

$$\hat{P}^- \zeta_s = -\eta .$$

As it can be seen, the solution (ζ_s, η) of this equation is a solution of system (27) at the limit of infinitely slow processes ($\partial^2 \eta / \partial t^2 \rightarrow 0$, $\partial Q / \partial t \rightarrow 0$) with sufficiently large horizontal scales ($\nabla^2 \zeta \rightarrow 0$).

The described filtering scheme is optimal in the sense it gives exact solutions for infinitely slow processes. For finite speed processes the filtered model yields approximate solutions, of course. Meanwhile, these solutions can be further improved, if needed, and the filtering scheme shows a natural way for the improvement. Let us write (27a) as a linear operator equation for ζ

$$\hat{P}\zeta = -A ,$$

where the source term A is a known function of η and Q_ζ . Then the nonfiltered equation (24a) can be represented as

$$\hat{P}\zeta = -A + \varepsilon \frac{\partial^2 \zeta}{\partial t^2} .$$

The last term on the right hand side can be treated for slow processes as a perturbation with the small perturbation parameter $\varepsilon = H_0^2 / c_a^2$, and the solution of this equation can be presented as a series

$$\zeta = \zeta_0 + \zeta_1 + \zeta_2 + \dots$$

where $\zeta_i \sim \varepsilon^i$. The first term ζ_0 represents a solution of the filtered equation (27a). Other members of the series are successive correction terms, which can be calculated from equations

$$\hat{P}\zeta_i = \varepsilon \frac{\partial^2 \zeta_{i-1}}{\partial t^2} , i = 1, 2, \dots .$$

Most meaningful is the first correction term, ζ_1 , which can be interpreted as an acoustic component of the motion, generated by slow dynamics.

4.2. The linear filtered dynamics. The linear model, corresponding to the filtered wave equations (27), can be derived from (21) by substituting equation (21a) with the balance condition

$$\hat{P}^+ w - b + \frac{Q}{T_0} = 0 . \tag{28}$$

This relation presents a diagnostic equation for ζ . The explicit equation for ζ can be obtained by differentiating (28) by t ; the result is the Poisson equation (27a). After this equation for ζ is employed, relation (28) can be used instead of (21c) for the determination of the vertical wind w .

As (28) is deduced from the initial linearized system in the limit $\partial\zeta/\partial t \rightarrow 0$, it can be treated as a consequence of the equation (18) and

$$\frac{w}{H_0} = -\frac{\omega}{p}. \quad (29)$$

Thus, the used filtration scheme employs the same relationship between w and ω as do the WM and MPM, and as a consequence, like these models it eliminates surface pressure waves (in accordance with the detailed balance on the right side of (17b), there is no lower boundary evolution in the filtered model). Equations (21b) – (21d) along with diagnostic relations (28) and (29) represent the closed linear acoustically filtered system of equations which is most close to initial nonfiltered linear model and which yields filtered wave–equations (27). Therefore, this set of equations will be a basis for nonlinear generalization.

The energy density of the filtered model (21b) – (21d), (28), (29) is

$$e_l = \frac{1}{2} \left(\frac{g^2}{N^2} \eta^2 + \mathbf{v}^2 + w^2 \right).$$

4.3. Compressibility in filtered model. Relation (29) presents the adjusted version of equation (16a) and coincides with the WM equation (12a). That means, the vertical velocity w is, as in the WM and MPM, an approximated field. Still, differently from the anelastic models WM and MPM, the present model preserves the thermodynamic relationship for ω , (18). This is achieved due to the maintenance of the compressibility. For the three–dimensional divergence of velocity in pressure–space,

$$D_p \equiv b + \frac{\partial\omega}{\partial p}, \quad (30)$$

from (18) and (29) a diagnostic equation follows

$$D_p = -\frac{N^2}{g}w + \frac{Q}{T_0}. \quad (31)$$

The right hand term is in general different from zero and the medium is compressible in p -space.

Comparison of equation (21b) with (30), (31) exhibits that η satisfies equation

$$\frac{\partial \eta}{\partial t} - b - \frac{\partial \omega}{\partial p} = 0 .$$

Because the density fluctuation, n' , satisfies the continuity equation (16e) (this is a matter of the definition of continuous medium), the sum $\eta_0 = n' + \eta$ presents a local invariant, which is constant in time at every point of the medium. This detailed balance of entropy and density fluctuations is the mechanism that eliminates the acoustic waves.

5. Nonlinear extensions of the filtered model

One possibility for nonlinear generalization is proposed by Rõõm and Ülejõe (1996). It consists of straightforward complementation of the system with the nonlinear continuity equation along with the substitution of local time derivative $\partial/\partial t$ by the construct nd/dt everywhere. Unfortunately, the energy conservation of the deduced in this way filtered nonlinear model is similarly to the MPM restricted to the case of conservative N . In nonlinear case this means simply that N is a constant. In addition, it is not clear whether the model supports the potential vorticity conservation or not.

To avoid these shortcomings, we propose an alternative way for nonlinear extension. Though not so "straightforward", it is quite general along with ideological transparency and simplicity, and it supports further generalizations (to the latitude-dependent f , spherical geometry, etc.). The method consists in nonlinear extension of the Hamiltonian principle¹⁾ for the wave-subsystem rather than the direct generalization of wave equations. The main idea of the method is that in the slow-moving medium with given velocity field $\{\mathbf{v}, \omega\}$ the wave equations are in the coordinate system which is bound to an individual air particle of the same form, as they would have at the resting background (ie., in the linear case) in the fixed coordinate system. It doesn't matter whether

1) The method can be equally formulated in terms of the least action principle, which would lead to nonlinear Lagrangian equations, second order in time. Still, the Hamiltonian principle is more preferable in nonlinear case as it gives first order equations.

the velocity field is independent or depends in turn on field variables. In other words, as first approximation we assume that in a slow flow the wave disturbances are carried along with the medium. The formal consequence of this assumption is that the local derivative $\partial/\partial t$ is replaced everywhere by the material derivative. Still, this is not made directly in wave equations but in the variational integral. In addition, \mathcal{H} is treated as the mass–density of energy (the wave energy of the unit mass), which permits to take the compressibility of the medium into account.

In detail the nonlinear version of the Hamiltonian principle for the nonfiltered wave system is discussed in Appendix B, and nonlinear, nonfiltered Hamiltonian wave equations are (B.3). To get the filtered version, the first equation (B.3a) must be omitted and the left hand term of (B.3b) must equal a zero. If we return in addition from the generalized momentum π_η back to the vertical velocity w with the help of (A.4b), the nonlinear, sound–relaxed wave equations are

$$\frac{d\eta}{dt} = -\frac{N^2}{g}w + \frac{Q}{T_0}, \quad (32a)$$

$$\frac{dw}{dt} = g\left(\hat{P}^-\zeta + \eta\right) - \left(w - \frac{gQ}{N^2T_0}\right)\frac{d}{dt}\ln\left(1 - f^2/N^2\right), \quad (32b)$$

$$H_0^2\left(\nabla \cdot n\nabla - nf^2/c_a^2\right)\zeta + \hat{P}^+\left\{n\left[\hat{P}^-\zeta + \left(1 - f^2/N^2\right)\eta\right]\right\} = -nQ\zeta. \quad (32c)$$

Remaining equations are the nonlinear potential vorticity equation

$$\frac{dq}{dt} = f\frac{\partial}{\partial p}\left(\frac{p}{T_i}Q\right), \quad (32d)$$

the continuity equation (3e), equation (29) for ω and diagnostical equations (22b), (25) and (28).

The energy of the model,

$$E = \int_{\mathcal{V}} n\mathcal{H}dx dy dp,$$

is conserved for $\mathcal{H} = \mathcal{T} + \mathcal{V}$, where \mathcal{V} is defined by formula (25c) and

$$\mathcal{T} = \frac{1}{2}\frac{N^2\pi_\eta^2}{1 - f^2/N^2} = \frac{1}{2}\left(N^2 - f^2\right)\left(\frac{w}{g} - \frac{Q}{N^2T_0}\right)^2.$$

The main differences of wave equations (32a) – (32c) in comparison with the linear case are (besides the substitution of the local time derivative by the material one) the

presence of the density n in the ζ -equation (32c) and the additional term in (32b). This term is required for energy conservation. In practice it is always small, because $f^2/N^2 \ll 1$, and it turns to exact zero for $f = 0$ and for constant f/N .

An interesting (and probably useful for practical applications) alternative to the derived system is the model with incompressible velocity field (with "incompressible advection"). Such a model can be developed, because the nonlinear Hamiltonian principle (B.2) does not depend explicitly on velocity field and supports models with incompressible material flow. The desired model follows from the previous, if n is put equal to unit in (32c), equation (3e) is replaced by the incompressibility condition (9) and equation (29) is left out. Relations (32a), (32b) and (32d) remain as they are. Equation (29) for ω , which is "thermodynamic" by its nature as the consequence of exact linear relation (18) is neglected in favour of the continuity condition (9). Supposedly equation (9) yields less accurate ω than (29). Still, the accuracy of the omega-velocity is not so crucial in the present approach, as it affects advective terms only and maintains linear terms untouched. Note also that although the material flow is incompressible, the model still represents a variety of elastic filtered models, because: (1) differently from the WM and MPM, equation (9) is not used for acoustic filtration; (2) the discussed model transforms at linearization back to the linear model (21).

6. Conclusions

The main aim of the present paper was to describe a general method for deriving filtered models of atmospheric dynamics. The method has the following advantages:

- The use of the Lagrangian formalism overcomes problems with energy conservation and concentrates attention on getting the best filtered models.
- As filtering is located in one scalar function – the Lagrangian – the possible filtering approximations are easy to control and classify. Consequently, the likelihood that some essential filtering scheme will be overlooked is small.
- The main attention is concentrated on the linear dynamics of the model. As the linear subsystem presents the backbone of every dynamic model of the continuous medium, it is most important to approximate the linear part of the model in an optimum way.

– The use of nonlinear extension of the Hamiltonian variational integral instead of the direct extension of wave equations is simultaneously simple and general ideologically and guarantees maintenance of symmetries of initial model in the final filtered version.

Though the developed filtering technique was realized on the example of pressure–space dynamics, it should work in common coordinates as well.

The practical output of the method is a filtered model, which has no previous analogues and which may be called with reference to the method of its derivation as the ”optimum filtered” model. This optimality in filtering does not automatically guarantee its quality, of course. Further investigation and comparison with other models and experiment are required for that.

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Appendix A: Hamiltonian equations for linear model

Generalized momenta for the Lagrangian (25) are

$$\pi_\zeta = \frac{\partial \mathcal{L}}{\partial \zeta_t} = \frac{\partial \mathcal{T}}{\partial \zeta_t} = \frac{H_0^2}{c_a^2} \zeta_t, \quad \pi_\eta = \frac{\partial \mathcal{L}}{\partial \eta_t} = \frac{\partial \mathcal{T}}{\partial \eta_t} = \left(1 - \frac{f^2}{N^2}\right) \frac{1}{N^2} \eta_t. \quad (A.1)$$

Using these definitions, kinetic energy (25b) can be presented as a function of π_ζ, π_η

$$\mathcal{T} = \frac{1}{2} \left(\frac{c_a \pi_\zeta}{H_0} \right)^2 + \frac{1}{2} \frac{N^2 \pi_\eta^2}{1 - f^2/N^2}$$

and the Hamiltonian (26) becomes a function of $\zeta, \eta, \pi_\zeta, \pi_\eta$.

The Hamiltonian principle is a variational extremum condition in the form

$$\delta \int_{t_0}^{t_1} dt \int_V dx dy dp (\zeta_t \pi_\zeta + \eta_t \pi_\eta - \mathcal{H}) = 0, \quad \forall \delta \zeta, \delta \eta, \delta \pi_\zeta, \delta \pi_\eta. \quad (A.2)$$

where variations of all fields must be zero at the initial and final moments and variations of ζ, η must be zero at the boundary of the domain V . Solutions of this extremum problem are the Hamiltonian equations

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \mathcal{H}}{\partial \pi_\zeta}, \quad \frac{\partial \eta}{\partial t} = \frac{\partial \mathcal{H}}{\partial \pi_\eta},$$

$$\begin{aligned}\frac{\partial \pi_\zeta}{\partial t} &= -\frac{\partial \mathcal{H}}{\partial \zeta} + \frac{\partial}{\partial x} \frac{\partial \mathcal{H}}{\partial \zeta_x} + \frac{\partial}{\partial y} \frac{\partial \mathcal{H}}{\partial \zeta_y} + \frac{\partial}{\partial p} \frac{\partial \mathcal{H}}{\partial \zeta_p} \equiv -\frac{\delta \mathcal{H}}{\delta \zeta}, \\ \frac{\partial \pi_\eta}{\partial t} &= -\frac{\partial \mathcal{H}}{\partial \eta} + \frac{\partial}{\partial x} \frac{\partial \mathcal{H}}{\partial \eta_x} + \frac{\partial}{\partial y} \frac{\partial \mathcal{H}}{\partial \eta_y} + \frac{\partial}{\partial p} \frac{\partial \mathcal{H}}{\partial \eta_p} \equiv -\frac{\delta \mathcal{H}}{\delta \eta}.\end{aligned}$$

Because

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial \pi_\zeta} &= \frac{\partial \mathcal{T}}{\partial \pi_\zeta} = \frac{c_a^2}{H_0^2} \pi_\zeta, \quad \frac{\partial \mathcal{H}}{\partial \pi_\eta} = \frac{\partial \mathcal{T}}{\partial \pi_\eta} = \frac{N^2 \pi_\eta}{1 - f^2/N^2}, \\ \frac{\delta \mathcal{H}}{\delta \zeta} &= \frac{\delta \mathcal{V}}{\delta \zeta} = \left(\frac{f H_0}{c_a} \right)^2 \zeta - H_0^2 \nabla^2 \zeta - \hat{P}^+ \left[\hat{P}^- \zeta + (1 - f^2/N^2) \eta \right] - Q_\zeta, \\ \frac{\delta \mathcal{H}}{\delta \eta} &= \frac{\delta \mathcal{V}}{\delta \eta} = (1 - f^2/N^2) \left(\hat{P}^- \zeta + \eta - Q_\eta \right),\end{aligned}$$

an explicit form of Hamiltonian equations is

$$\frac{\partial \zeta}{\partial t} = \frac{c_a^2}{H_0^2} \pi_\zeta, \quad \frac{\partial \eta}{\partial t} = \frac{N^2 \pi_\eta}{1 - f^2/N^2}, \quad (\text{A.3a})$$

$$\frac{\partial \pi_\zeta}{\partial t} = H_0^2 (\nabla^2 - f^2/c_a^2) \zeta + \hat{P}^+ \left[\hat{P}^- \zeta + (1 - f^2/N^2) \eta \right] + Q_\zeta, \quad (\text{A.3b})$$

$$\frac{\partial \pi_\eta}{\partial t} = - (1 - f^2/N^2) \left(\hat{P}^- \zeta + \eta - Q_\eta \right). \quad (\text{A.3c})$$

Elimination of the generalized momenta from these equations gives the Lagrangian wave equations (24). An acoustically filtered variant follows, if $\partial \zeta / \partial t$ and π_ζ are put to zero.

A comparison of (A.3a) with (21a) and (21b) yields relationships between generalized momenta and common field variables:

$$\pi_\zeta = \frac{H_0^2}{\alpha c_a^2} \left(\frac{1}{H_0} \hat{P}^+ w - b + \frac{Q}{T_0} \right), \quad (\text{A.4a})$$

$$\pi_\eta = (1 - f^2/N^2) \left(-\frac{w}{g} + \frac{Q}{N^2 T_0} \right). \quad (\text{A.4b})$$

These equations enable to find fields b and w after the Hamiltonian system (A.3) is integrated. Left side of (A.4a) makes zero for adjusted model and this equation transforms to (28).

Appendix B: Nonlinear Hamiltonian equations for the wave-subsystem

Here we generalize the Hamiltonian formalism of Appendix A to the moving continuous medium with given slow velocity field $\mathbf{v}(\mathbf{x}, p, t)$, ω , i.e. for a given trajectory ensemble of infinitesimal fluid particles. Though this generalization is required for acoustically relaxed dynamics, we deduce Hamiltonian equations for general non-filtered wave system as the method works equally correctly for the nonfiltered model as well.

The generalization rests on the following assumptions.

1. The density n follows for given \mathbf{v} the continuity equation

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) + \frac{\partial n\omega}{\partial p} = 0, \quad (B.1)$$

Density n is here entirely determined by the given flow pattern. A special case is the volume-preserving flow field,

$$\nabla \cdot \mathbf{v} + \frac{\partial \omega}{\partial p} = 0, \quad (B.1')$$

at which $n = 1$.

2. The wave energy density \mathcal{H} is the mass density: $n\mathcal{H}dx dy dp$ represents the energy of an individual fluid particle.

3. The trajectories and locations of particles are given and not subjected to variation. The independent variable fields are, as in linear wave-subsystem, $\eta, \zeta, \pi_\eta, \pi_\zeta$, but their independent variations are carried out in individual, moving fluid particles rather than in fixed p -space points. Consequently, the time derivatives of field variables correspond to the fluid particles rather than to fixed points, which means that the partial derivative $\partial/\partial t$ is replaced by material derivative d/dt in the Hamiltonian principle formulation.

At made assumptions the generalization of the Hamiltonian principle (A.2) reads

$$\delta \int_{t_0}^{t_1} dt \int_V dx dy dp n \left(\pi_\zeta \frac{d\zeta}{dt} + \pi_\eta \frac{d\eta}{dt} - \mathcal{H} \right) = 0, \quad \forall \delta\zeta, \delta\eta, \delta\pi_\zeta, \delta\pi_\eta, \quad (B.2)$$

where variations of all fields must be zero at the initial and final moments and variations of ζ, η must be zero at the boundary of the domain V .

Due to the continuity equation we have

$$\int_{t_0}^{t_1} dt \int_V dx dy dp n \frac{d\delta\varphi_1}{dt} \varphi_2 = - \int_{t_0}^{t_1} dt \int_V dx dy dp n \frac{d\varphi_2}{dt} \delta\varphi_1$$

for every $\delta\varphi_1$, φ_1 , if $\delta\varphi_1$ is zero at the boundaries of the domain V . Keeping this in mind, we get from (B.2) the explicit Hamiltonian equations

$$\frac{d\zeta}{dt} = \frac{c_a^2}{H_0^2}\pi_\zeta \quad , \quad \frac{d\eta}{dt} = \frac{N^2\pi_\eta}{1 - f^2/N^2} \quad , \quad (B.3a)$$

$$n\frac{d\pi_\zeta}{dt} = H_0^2 (\nabla \cdot n\nabla - nf^2/c_a^2)\zeta + \hat{P}^+ \left\{ n \left[\hat{P}^-\zeta + (1 - f^2/N^2)\eta \right] \right\} + nQ_\zeta \quad , \quad (B.3b)$$

$$\frac{d\pi_\eta}{dt} = - (1 - f^2/N^2) \left(\hat{P}^-\zeta + \eta - Q_\eta \right) \quad . \quad (B.3c)$$

In comparison with the resting-medium model discussed in Appendix A, the time derivatives here are the Lagrangian derivatives and equation (B.3b) includes density n . Relationships (A4) remain unchangeable.

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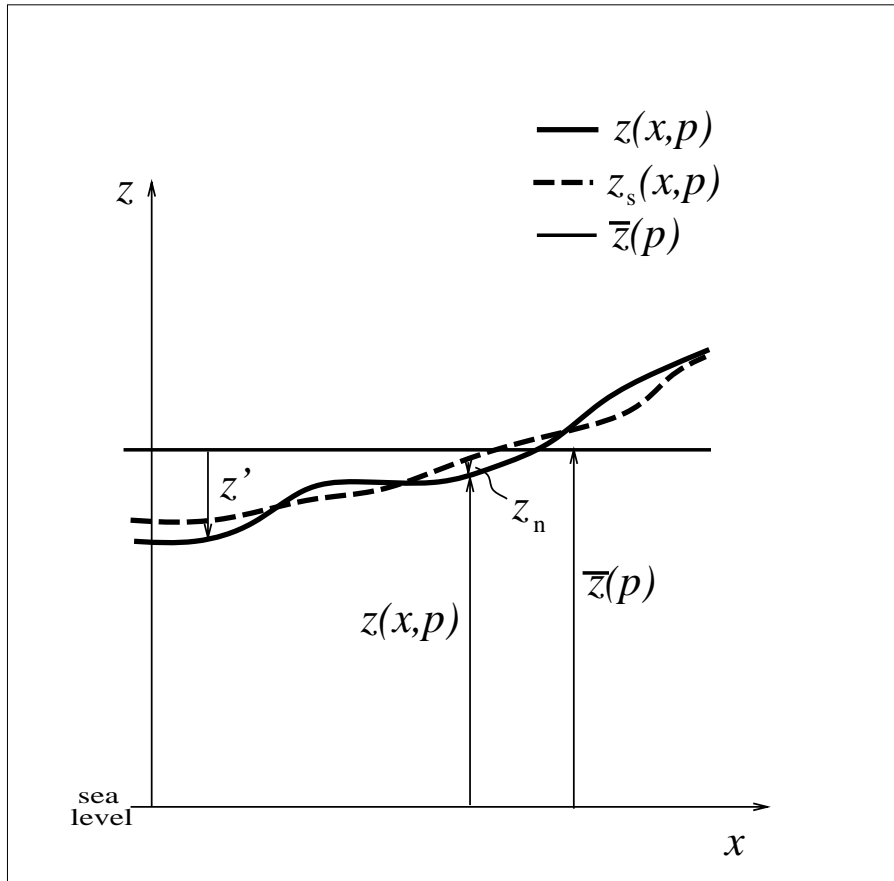


Fig. 1

Heights z , z_s and \bar{z} as functions of the horizontal coordinate x of isobaric surfaces $p = \text{const.}$, corresponding to the actual, hydrostatic and mean pressure distributions in the atmosphere.

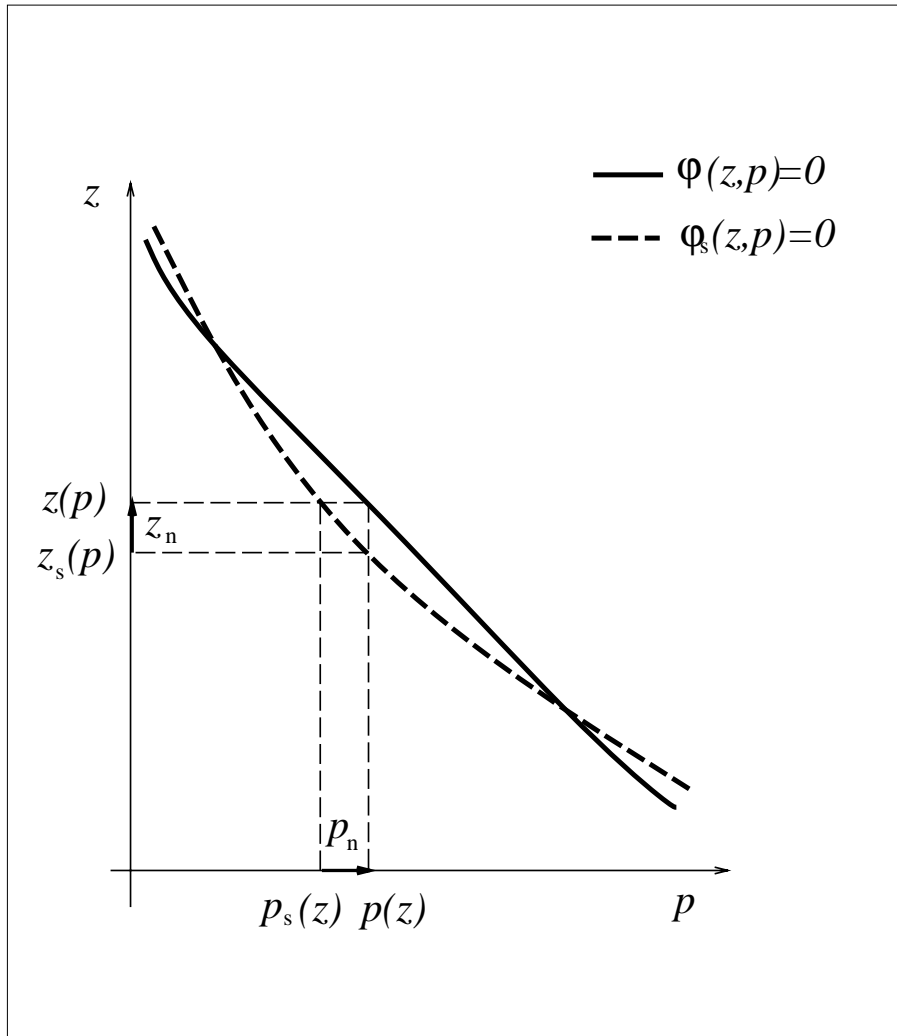


Fig. 2

An illustration to relation (1c). φ and φ_s represent the actual and the hydrostatic implicit relationships between height z and pressure p for fixed t and \mathbf{x} .