Q. J. R. Meteorol. Soc. (1998), **999**, pp. 1–999

LEAST ACTION PRINCIPLE FOR GENERAL, NONHYDROSTATIC, COMPRESSIBLE, ACOUSTICALLY NON-FILTERED PRESSURE-COORDINATE MODEL

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(Received 23 June 1997; revised 10 May 1998)

SUMMARY

The least action principle is described for general (nonhydrostatic, compressible, acoustically non-filtered) pressure–coordinate equations of atmospheric dynamics (Rõõm 1989), which represents a generalization of the Least Action Principle, first developed for the pressure-coordinate representation of atmospheric dynamics by Salmon and Smith (1994).

KEYWORDS: General hydrodynamical equations, Isobaric coordinates, Variational principles, Acoustical relaxation

1. INTRODUCTION

Beginning with classical works by Serrin (1959, henceforth S59) and Arnol'd (1965, 1969), the application of variational principles in atmospheric dynamics has been attracting attention of many investigators (Salmon 1983, 1988, Shepherd 1990, Salmon and Smith 1994 (further SS94), Roulstone and Brice 1995, Rõõm 1998a). Apart from the intrinsic aesthetic appeal, the use of the least action principle (LAP) for foundation of hydrodynamic models is also of practical importance at least in two respects. First, the existence of variational formulation exhibits agreement of the model with fundamental laws of nature. As it is proved in *Noether Theorem*, the variational formulation of dynamics is automatically accompanied with the existence of integrals of motion, the number and quality of which is uniquely determined by temporal and spatial symmetries of the action integral. Secondly, the variational method provides powerful tools for model development. It is safe and straightforward to employ the variational formulation for deduction of different simplified sub-models which are easier to study or integrate numerically and which still maintain the desired symmetries and conservative qualities of the initial, nonsimplified model. Keeping this in mind, it is advantageous to have variational formulation for pressure–coordinate models, too. The first variational formulation of atmospheric dynamics in pressure coordinates is presented by Salmon and Smith in the important paper SS94 for the White (1989) anelastic pressure–coordinate model. As it turns out, it is quite straightforward to generalize results of the SS94 to include all dynamical effects present in the atmosphere and to get a variational principle for general atmospheric dynamics in pressure coordinates without filtering simplifications. In the present note, we describe the required generalization. The resulting LAP describes the general, compressible, acoustically nonfiltered, pressure–coordinate dynamics (Rõõm 1989, 1990). Different filtered models, including the SS94, represent approximations which can be achieved, introducing necessary modifications into the action integral.

2. THE LAP FOR GENERAL, ACOUSTICALLY NONFILTERED PRESSURE-COORDINATE DYNAMICS

The required generalization of the SS94 results can be obtained, **(1)** retaining the full vertical velocity $\dot{z}$ in the Lagrangian action density, and **(2)**, introducing explicitly the continuity equation for the mass density n in pressure coordinates. The last modification can be easily introduced following the LAP formulation of the S59 (except that Cartesian coordinates, used in the S59 are replaced by pressure–coordinates).

We define the action integral

$$S = \int_{t_0}^{t_1} dt \int_V n\, l(z,\ \dot{z},\ \dot{x},\ \dot{y},\ p,\ n,\ s)\ dV\ , \tag{1}$$

where $dV = dxdydp$ is the volume element in pressure coordinates, n represents the nondimensional density, defined via the mass element $dM = ndV/g$, and g is the constant gravity acceleration. l represents the Lagrangian density of action, which for general, nonfiltered case can be presented as

$$l(z,\ \dot{z},\ \dot{x},\ \dot{y},\ p,\ n,\ s) = \frac{\dot{x}^2}{2} + \frac{\dot{y}^2}{2} + \frac{\dot{z}^2}{2} - h(p,s) - gz\cdot\left(1 - \frac{1}{n}\right)\ ,$$

where $h(p,s)$ represents the enthalpy as the known thermodynamic characteristic function of pressure p and entropy s, for which

$$\left(\frac{\partial h}{\partial p}\right)_s = \frac{1}{\rho}\ , \quad \left(\frac{\partial h}{\partial s}\right)_p = \theta\ ,$$

ρ ant θ being the density and temperature.

Variational principle for action (1) is

$$\delta S = 0 \quad \forall \quad \{\delta x(x,y,p,t)\ , \delta y(x,y,p,t)\ , \delta p(x,y,p,t)\ ,\ \delta z(x,y,p,t)\}\ . \tag{2}$$

Here δx δy, δp and δz are fields of independent variations (virtual replacements) of coordinates x, y, p and isobaric height z of air particles. It is assumed, that these replacements are differentiable functions, which become zero at boundaries of the domain V and in the initial and final instants t_0 and t_1. Note that the number of independently varied fields in pressure space is larger by one in comparison with the common Cartesian-coordinate formulation (S59). The four–dimensional nature of the field of independent variations is the main quality of the pressure–coordinate formulation which cannot be deduced in a trivial manner with the analogy to the common–space formulation.

Additional constrictions, laid on the variations of S, are similar to the S59 and are as follows.

(a) The elementary mass is conserved in pressure–space at virtual replacements: $\delta\,(dV\cdot n) = 0$, which yields for the density variation

$$\delta n + n\left(\frac{\partial}{\partial \mathbf{x}}\cdot\delta\mathbf{x} + \frac{\partial \delta p}{\partial p}\right) = 0\ . \tag{3}$$

(b) The entropy is conserved at variations

$$\delta s = 0\ . \tag{4}$$

(c) The virtual replacements involve the trajectories (both in the ordinary physical and pressure–coordinate spaces), from which the commutation rules arise:

$$\delta\dot{x} = \frac{\mathrm{d}}{\mathrm{d}t}\delta x\ ,\quad \delta\dot{y} = \frac{\mathrm{d}}{\mathrm{d}t}\delta y\ ,\quad \delta\dot{z} = \frac{\mathrm{d}}{\mathrm{d}t}\delta z\ ,\quad \delta\dot{p} = \frac{\mathrm{d}}{\mathrm{d}t}\delta p\ . \tag{5}$$

3. Equations of motion

Equations of motion are extremes of variational problem (2) at additional restrictions (3) – (5). We represent the final result without details (which can be found in Rõõm 1998b. In general, the deduction follows S59). The four equations, corresponding to four independent variations, are

$$\delta z: \quad n\,\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} \;=\; g(1\;-\;n)\;, \tag{6a}$$

$$\delta x: \quad n\,\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} \;=\; -\;g\frac{\partial z}{\partial x}\;, \tag{6b}$$

$$\delta y: \quad n\,\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} \;=\; -\;g\frac{\partial z}{\partial y}\;, \tag{6c}$$

$$\delta p: \quad \frac{\partial z}{\partial p} \;=\; -\,\frac{n}{g}\,\frac{\partial h(p,s)}{\partial p}\;. \tag{6d}$$

(For simplicity we have dealt with non-rotating coordinates and the Coriolis force is absent). This set of equation must be complemented with the continuity and entropy equations, which follow for real motion from (3) and (4) in the form

$$\frac{\mathrm{d}n}{\mathrm{d}t} \;+\; n\left(\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} \;+\; \frac{\partial \dot{p}}{\partial p}\right) \;=\; 0\;, \tag{6e}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} \;=\; A_s\;. \tag{6f}$$

System (6) coincides with the equations, derived by Rõõm (1989, 1990).

4. Acoustic filtering in Lagrangian function

The variational formulation has practical value as an efficient tool for getting different simplified models which maintain the essential qualities (like symmetries and consequent conservation laws) of the initial, nonsimplified dynamics. In particular, different acoustically filtered pressure–coordinate models, which are of great interest for numerical applications, can be deduced, compared and studied in the unified framework. As the existing filtered models (Miller and Pearce 1974, White 1989, SS94) as well as different filtering schemes in pressure coordinates (Rõõm 1998a) demonstrate, the approximation that effectively filters acoustic component but maintains slow dynamics almost unchanged, is the one which approximates the vertical velocity $\dot{z}$ by the material derivative of an appropriately chosen function of other fields and independent variables:

$$\dot{z} \approx \frac{\mathrm{d}}{\mathrm{d}t}\phi(x,y,p,t,s,n,...)\;, \tag{7}$$

and which yields approximation of the Lagrangian density by function

$$\tilde{l}(z,\dot{x},\dot{y},p,\dot{p},n,s,t) \;=\; l[z,\;\dot{\phi}(x,y,p,t,s,n,..),\dot{x},\dot{y},p,n,s]\;.$$

The three known special cases of this general approximation are:

(i) The nonhydrostatic anelastic model by Miller and Pearce:

$$\phi = z_0(p) \quad\rightarrow\quad \dot{z} \approx \omega\frac{\partial z_0}{\partial p} = -\frac{\omega}{g\rho_0}\;, \tag{8}$$

where z_0 represents the horizontally homogeneous background distribution of the hydrostatic isobaric height and $\rho_0(p)$ is the corresponding background density. This approximation has been used in numerical models by Miranda and James (1992) and Miranda and Valente (1997) and has proved an effective approximation in meso-scale modeling.

(ii) Approximation introduced by Salmon and Smith (SS94)

$$\phi = -\frac{h(p,s)}{g} \quad \rightarrow \quad \dot{z} \approx -\frac{\omega}{g\rho} - \frac{T\dot{s}}{g} \ . \tag{9}$$

This approximation represents a generalization of the White model (1989) and yields the last one for the conservative case, $\dot{s} = 0$.

(iii) Hydrostatic approximation

$$\phi \equiv 0 \quad \rightarrow \quad \dot{z} \approx 0 \ . \tag{10}$$

Though this approximation represents the long–wave asymptote of the two previous cases, it can be introduced promptly into the Lagrangian density l to get the hydrostatic primitive equation model as the extremum of variational problem.

As a special case of (7), not considered elsewhere, and generalizing the Miller and Pearce model, is the approximation

$$\phi = z_s(x,y,p) \quad \rightarrow \quad \dot{z} \approx \frac{\mathrm{d}z_s}{\mathrm{d}t} \ . \tag{11}$$

Here $z_s(x,y,p)$ is an appropriately chosen isobaric height distribution, not necessarily homogeneous in horizontal (a temporal mean of the actual air-mass, for instance).

The common quality of all described filtering approximations (8) – (10) is the absence of time argument t in the definitions of ϕ. When this restriction is not maintained and ϕ includes t among arguments, the approximated Lagrangian will include t explicitly, which will cause in accordance with *Noether Theorem* the energy conservation violence.

The acoustic wave filtering by (7) is a general quality, inherent to this approximation, and does not depend on the special choice of ϕ. Once (7) is introduced into the Lagrangian, the variation of action S in z yields the anelastic condition for the pressure–coordinate density: $n = 1$. As a consequence, the continuity equation (6e) reduces to

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{p}}{\partial p} = 0 \ , \tag{12}$$

which yields the non-divergent nature of three-dimensional flow and filtering of motions connected with volume changes in pressure-coordinates. As in the special case of hydrostatic approximation the acoustic disturbances are eliminated, one can conclude that this is the case for all models which are based on approximation (7).

Detailed derivation of filtered models using approximations of the action S is presented in (Rõõm 1998b). Explicit representation of model equations corresponding to approximation (8) can be found in (Miller and Pearce 1974), the model based on (9) is presented by White (1989) and in the SS94. Here we present the model resulting from approximation (11), which has not been formerly published. Defining the reference density ρ_s and the vertical velocity w_s according to relations

$$\frac{\partial z_s}{\partial p} = -\frac{1}{g\rho_s} \ ,$$

$$\frac{\mathrm{d}z_s}{\mathrm{d}t} = w \ ,$$

the analogues of equations of motion (6a) – (6c), representing the extremal of the approximated action, are

$$\frac{\mathrm{d}w}{\mathrm{d}t} = g^2\rho_s\left(\frac{1}{g\rho} + \frac{\partial z}{\partial p}\right) ,$$

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -g\nabla z - \frac{\mathrm{d}w}{\mathrm{d}t}\nabla z_s .$$

where $\mathbf{v} = \{\dot{x}, \dot{y}\}$ is the horizontal velocity. This model becomes closed after supplementation with the equation (12) and (6e).

Acknowledgements

This investigation has been supported by the Estonian Science Foundation under Grant No. 2624.

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