NON–HYDROSTATIC ACOUSTICALLY FILTERED EQUATIONS OF ATMOSPHERIC DYNAMICS IN PRESSURE COORDINATES

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Abstract. Non-hydrostatic acoustically filtered equations of motion of nonviscid fluid are derived in pressure coordinates. Complete set of non–hydrostatic nonlinear equations for ideal fluid in pressure coordinates serves as starting basis. These equations are linearized and transformed to a convenient for filtering form. Acoustic filtering is achieved in the limit of infinitely high sound speed, $c_a \rightarrow \infty$. The filtered model lacks acoustic wave solutions but maintains without loss of accuracy all slow processes, including buoyancy waves. The obtained in this way linear model is complemented to a nonlinear set by inclusion of incompressible advection terms in pressure space. The final equations are capable of describing slow processes from local turbulence to planetary–scale waves. Still, the main domain of application of the model is mesoscale dynamics.

1. INTRODUCTION

The idea to use pressure related coordinates for non–hydrostatic (NH) dynamics is not new. A pressure coordinate (p–coordinate) acoustically filtered NH model was first proposed by Miller [1] and Miller & Pearce [2]. The Miller-Pearce model (MPM) has been widely used in numeric modeling [3 − 5]. A variant of the numeric package, developed in [5], is presently in use at Tartu Observatory. Though the p–coordinate presentation in combination with the non–hydrostatic assumption looks exotic and sophisticated at first encounter, its incontestable advantage consists in ability to treat non–hydrostatic processes of shorter mesoscale ($l_x \sim 10^2 - 10^3$ m) and hydrostatic processes of longer mesoscale ($l_x \sim 10^4 - 10^5$ m) and synoptic scale ($l_x \sim 10^6 - 10^7$ m) in the framework of unified formalism.

The main assumption of the MPM is the approximation of incompressibility of motion in p–space, which filters sound waves:

$$\nabla \cdot \mathbf{v} + \frac{\partial \omega}{\partial p} = 0 . \qquad (I1)$$

Here $\mathbf{v}$ and $\omega = \mathrm{d}p/\mathrm{d}t$ are the horizontal wind vector and the vertical speed of the air–particle in the p–space. Assumption (I1) enables to filter acoustic waves, unwanted in slow dynamics, in a most straightforward manner. As the same assumption is exact in the hydrostatic limit, (I1) presents an extrapolation of the main characteristic feature of primitive equations to the mesoscale.

Here we introduce a different acoustically filtered NH model, which rejects hypotheses (I1). Though approximation (I1) will be used at the final stage of the model development for the introduction of nonlinear momentum advection, it is not used for wave filtering.

The model we develop is based on the complete set of NH equations in p–coordinates [6]. The filtering proceeds as follows:
At first the initial set of non–hydrostatic p–coordinate equations is linearized. The filtering is carried out in the linear model. Finally, the obtained model is complemented to a nonlinear filtered model with maintenance of energy conservation.

2. LINEAR NH MODEL IN p–COORDINATES

2.1. Linearization of equations of paper [6] according to the hydrostatic equilibrium state, characterized by the mean temperature, $T_0(p)$, yields equations

$$\frac{\partial z'}{\partial t} = v_z + H_0 \frac{\omega}{p} , \qquad (1a)$$

$$\frac{\partial v_z}{\partial t} = -g^2 \rho' , \qquad (1b)$$

$$\frac{\partial \mathbf{v}}{\partial t} \;=\; -\; g\nabla\; z'\;, \tag{1c}$$

$$\frac{\partial T'}{\partial t} \;=\; \frac{T_i \omega}{p} \;+\; Q\;, \tag{1d}$$

$$\frac{\partial \rho'}{\partial t} \;=\; -\;\frac{1}{g}(\nabla\cdot\mathbf{v} \;+\; \partial\omega/\partial p)\;, \tag{1e}$$

$$\frac{\partial p'_s}{\partial t} \;=\; \omega|_{p_0}\;, \tag{1f}$$

$$\rho' \;=\; -\;\frac{1}{g}\left(\frac{p}{H_0}\frac{\partial z'}{\partial p} \;+\; \frac{T'}{T_0}\right)\;. \tag{1g}$$

Here $v_z \;=\; \mathrm{d}z/\mathrm{d}t$ is vertical velocity, z' and T' represent isobaric height and temperature fluctuations, $\rho' = \rho - 1/g$ is the p–space density fluctuation from its equilibrium constant value $\overline{\rho} = 1/g$, and p'_s is the ground surface pressure fluctuation from the mean pressure distribution at the ground p_0. Parameter $H_0 \;=\; RT_0/g$ presents the height scale for hydrostatic pressure,

$$T_i \;=\; \frac{R}{c_p}T_0 \;-\; p\frac{\partial T_0}{\partial p}$$

is the stability parameter ("stability temperature") of the background state and Q represents the given thermal forcing.

2.2. Boundary conditions. Conditions at the lateral boundaries are the same as in cartesian coordinate models and do not present special interest in the context of the present study. The main differences with ordinary model occur in the "horizontal" conditions at the top and at the bottom. The domain occupied by the atmosphere in the p–space is

$$0 \;<\; p \;< p_0(\mathbf{x},\; t)\;, \quad -\,\infty \;<\; x,\; y \;<\; \infty\;, \tag{2}$$

Boundary conditions at the bottom and top are

$$z'|_{p_0} \;=\; H_0(p_0)\frac{p'_s}{p_0}\;, \quad v_z|_{p_0} \;=\; \mathbf{v}|_{p_0}\cdot\nabla h\;, \quad \omega|_0 \;=\; 0\;, \tag{3}$$

where $h(\mathbf{x})$ is the ground surface height above the sea–level.

2.3. Diagnostic equation for ω. Model (1) presents a closed system consisting of eight equations for eight fields z', v_x, v_y, v_z, T', ρ', p'_s and ω. All quantities here, except ω, are prognostic fields, and system (1) includes a single diagnostic equation (1f). This equation should be used for the determination of the diagnostic field ω. As (1f) does not include ω explicitly, the only way to proceed is to differentiate (1f) by t and eliminate time derivatives by the help of other equations. The result is an explicit equation for ω

$$\alpha\;\frac{\omega}{p} \;=\; \frac{p}{H_0}\;\frac{\partial v_z}{\partial p} \;-\; \nabla\cdot\;\mathbf{v} \;+\; \frac{Q}{T_0}\;, \tag{4}$$

where

$$\alpha \; = \; c_v / c_p \; .$$

This relation presents the linearized pressure tendency equation in p–coordinate representation.

2.4. The reduced linear system. The obtained diagnostic equation (4) along with equation (1g) enables to get from (1) a reduced set of equations which is closed according to z', v_z, $\mathbf{v}$, T' and p'_s and does not include ρ' and ω (though equations for these fields, (1e) and (4), remain valid)

$$\frac{1}{c_a^2}\frac{\partial \zeta}{\partial t} \; = \; \frac{1}{gH_0}\left[\frac{1}{H_0}\left(\alpha \; + \; p\frac{\partial}{\partial p}\right) v_z \; - \; \nabla \cdot \mathbf{v} \; + \; \frac{Q}{T_0}\right] \; . \qquad (5a)$$

$$\frac{\partial \eta}{\partial t} \; = \; - \; \frac{T_i v_z}{T_0 H_0} \; + \; \frac{Q}{T_0} \; , \qquad (5b)$$

$$\frac{\partial v_z}{\partial t} \; = \; g\left[\left(\frac{\partial}{\partial p}p \; - \; \alpha\right)\zeta \; + \; \eta\right] \; , \qquad (5c)$$

$$\frac{\partial \mathbf{v}}{\partial t} \; = \; - \; gH_0 \nabla \; \zeta \; , \qquad (5d)$$

$$\frac{\partial p'_s}{\partial t} \; = \; \left[\frac{p}{\alpha}\left(\frac{p}{H_0}\,\frac{\partial v_z}{\partial p} \; - \; \nabla \cdot \; \mathbf{v} \; + \; \frac{Q}{T_0}\right)\right]_{p=p_0} \; . \qquad (5e)$$

We have introduced non–dimensional fluctuative fields in place of z' and T':

$$\zeta \; = \; \frac{z'}{H_0} \; , \qquad \eta \; = \; \frac{T'}{T_0} \; - \; \frac{T_i z'}{T_0 H_0} \; . \qquad (6)$$

2.5. Wave equations. It is easy to get two second order equations for ζ and η, differentiating (5a) and (5b) according to the time and eliminating the first order time derivatives with the help of (5c) and (5d):

$$\left[H_0^2\left(\frac{1}{c_a^2}\frac{\partial^2}{\partial t^2} \; - \; \nabla^2\right) \; - \; \left(p\frac{\partial}{\partial p} \; + \; \alpha\right)\left(\frac{\partial}{\partial p}p \; - \; \alpha\right)\right]\zeta \; - \; \left(p\frac{\partial}{\partial p} \; + \; \alpha\right)\eta \; = \; \frac{R}{g^2}\frac{\partial Q}{\partial t} \; , \qquad (7a)$$

$$\left(\frac{1}{N^2}\frac{\partial^2}{\partial t^2} \; + \; 1\right)\eta \; + \; \left(\frac{\partial}{\partial p}p \; - \; \alpha\right)\zeta \; = \; \frac{RT_0}{g^2 T_i}\frac{\partial Q}{\partial t} \; . \qquad (7b)$$

where $c_a \; = \; \sqrt{RT_0/\alpha}$ is the sound speed and $N = \sqrt{RT_i}/H_0$ represents the Väisälä frequency. These equations can be employed for modeling linear wave processes in p–coordinate presentation in a general, non–filtered case.

2.6. The Lagrangian function and energy. For the present study the significance of wave equations is that they have a Lagrangian function

$$\mathcal{L} \; = \; \frac{1}{2}\left\{ - \; \frac{H_0^2}{c_a^2}(\zeta_{,t})^2 \; - \; \frac{1}{N^2}(\eta_{,t})^2 \; + \; H_0^2(\nabla\zeta)^2 \; + \; \left[\left(\frac{\partial}{\partial p}p \; - \; \alpha\right)\zeta \; + \; \eta\right]^2\right\} \; . \qquad (8)$$

On the one hand, the existence of the Lagrangian guarantees energy conservation. On the other hand, with the help of Lagrangian formalism it is easy to get filtered versions of the model which are still energy–conserving. Explicitly the energy density can be presented for model (5) as

$$e \;=\; \frac{1}{2}\left(\frac{g^2 H_0^2}{c_a^2}\zeta^2 \;+\; \frac{RT_0^2}{T_i}\eta^2 \;+\; \mathbf{v}^2 \;+\; v_z^2\right) \;. \tag{9}$$

The first two terms present potential energy which air particle has due to the isobaric height and temperature fluctuations, the remaining two terms are kinetic energy.

3. ACOUSTIC FILTERING

For slow atmospheric movements with small Froude number,

$$\mathcal{F} \;\equiv\; U^2/c_a^2 \;<<\; 1 \;, \tag{10}$$

where U is the characteristic amplitude of the velocity, it is reasonable to simplify model equations in the way they do not include acoustic–wave solutions anymore. This procedure is called filtering.

3.1. Filtering of linearized equations (5).

The physical basis and proof for acoustic filtering can be received from the scale analysis of the Lagrangian (8). It is easy to verify that the first term in (8) is small in comparison with others in all scales if the Froude number is small and, thus, it can be neglected. Formally filtering is straightforward with the help of the limiting process

$$c_a \;\to\; \infty \tag{11}$$

in equation the (5a), which yields

$$\frac{1}{H_0}\left(\alpha \;+\; p\frac{\partial}{\partial p}\right)v_z \;-\; \nabla\cdot\mathbf{v} \;+\; \frac{Q}{T_0} \;=\; 0 \;. \tag{12}$$

This equation along with other equations (5b) – (5e) (which did not change at the filtering) presents the basic acoustically filtered set of linear equations in p–coordinates. The wave equations for the filtered model can be obtained from (7) with the help of the same formal filtering procedure (11) and they possess the Lagrangian function

$$\mathcal{L} \;=\; \frac{1}{2}\left\{-\;\frac{1}{N^2}(\eta,_t)^2 \;+\; H_0^2(\nabla\zeta)^2 \;+\; \left[\left(\frac{\partial}{\partial p}p \;-\; \alpha\right)\zeta \;+\; \eta\right]^2\right\} \;. \tag{8$'$}$$

As a consequence, filtering does not harm energy conservation. The energy density can be deduced from (9), using passage (11):

$$e \;=\; \frac{1}{2}\left(\frac{RT_0^2}{T_i}\eta^2 \;+\; \mathbf{v}^2 \;+\; v_z^2\right) \;. \tag{9$'$}$$

Equation (12) presents the main diagnostic relation in the filtered model. In non–stationary problems it presents the basic equation for ζ determination, simultaneously it can be used for calculation of the vertical speed v_z. After filtering is carried out, it is impossible to return from the field η back to the ordinary temperature fluctuation T' and the hydrodynamic content of η alters. In the filtered model it has the content of the relative density fluctuation. To prove this feature, we note that from equations (4) and (12) a relationship follows

$$\nabla \cdot \mathbf{v} + \frac{\partial \omega}{\partial p} = -\frac{T_i v_z}{T_0 H_0} + \frac{Q}{T} .$$

Comparison of this equation with (5b) exhibits that η satisfies condition

$$\frac{\partial \eta}{\partial t} - \nabla \cdot \mathbf{v} - \frac{\partial \omega}{\partial p} = 0 .$$

That means, η evolves according to the same equation which is valid for $-g\rho'$ in nonfiltered model.

3.2. Inclusion of advective processes into the filtered model can be achieved in the simplest manner by modeling them as quasi–incompressibles in the p–space. For that in adjusted model the local time derivatives $\partial/\partial t$ should be replaced by the individual derivatives

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \omega_s \frac{\partial}{\partial p} , \tag{13}$$

in momentum equations (5c) and (5d). Here ω_s is the vertical p–velocity of the incompressible flow:

$$\frac{\partial \omega_s}{\partial p} + \nabla \cdot \mathbf{v} = 0 . \tag{14}$$

At the same time, (5b) should be maintained in its initial linear form, as η, having the content of small relative density fluctuations, is not redistributed in space advectively. The resulting filtered equations are

$$\frac{1}{H_0} \left(\alpha + p \frac{\partial}{\partial p} \right) v_z - \nabla \cdot \mathbf{v} + \frac{Q}{T_0} = 0 , \tag{15a}$$

$$\frac{\partial \eta}{\partial t} = - \frac{T_i v_z}{T_0 H_0} + Q/T_0 , \tag{15b}$$

$$\frac{\mathrm{d} v_z}{\mathrm{d}t} = g \left[\left(\frac{\partial}{\partial p} p - \alpha \right) \zeta + \eta \right] , \tag{15c}$$

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = - g H_0 \nabla \zeta , \tag{15d}$$

$$\frac{\mathrm{d}p'_s}{\mathrm{d}t} = \left[\frac{p}{\alpha} \left(\frac{p}{H_0} \frac{\partial v_z}{\partial p} - \nabla \cdot \mathbf{v} + \frac{Q}{T_0} \right) \right]_{p=p_0} . \tag{15e}$$

Thus, our model treats linear processes as compressible in the p–space, while nonlinear processes are approximated as incompressible. The energy density for this nonlinear model coincides with the energy density of the linear acoustically filtered case (9').

CONCLUSIONS

We have developed a version of acoustically filtered set of model equations for atmospheric dynamics. Differently from common models, like the MPM in p–space or anelastic models in ordinary Cartesian coordinates, our model does not use incompressibility for wave filtering. Certainly, the quality of the model should be tested in futher experiments. Nevertheless, it is optimal at least in one respect: in linear case it presents the best approximation to the non–filtered equations.

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MITTEHÜDROSTAATILISED AKUSTILISELT FILTREERITUD ATMOSFÄÄRIDÜNAAMIKA VÕRRANDID RÕHUKOORDINAATIDES

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Käesolevas töös kirjeldatakse optimaalset akustiliselt filtreeritud atmosfääridünaamika mudelvõrrandite tuletusalgoritmi rõhukoordinaatides. Lähtutakse töös [6] tuletatud täielikest mittelineaarsetest hüdrodünaamika võrranditest rõhukoordinaatides. Need võrrandid lineariseeritakse ja tuuakse kujule (5), kus akustiline filtreerimine on lihtsaimal viisil teostatav piirüleminekuga lõpmata suurele häälekiirusele, $c_a \rightarrow \infty$. Tulemusena väheneb võrrandite ajaline järk kahe võrra ning kaovad häälelained. Saadud akustiliselt filtreeritud lineaarsed võrrandid täiendatakse advektiivsete liikmete lisamisega tagasi mittelineaarseteks. Seejuures lähendatakse advektsiooni mudeliga, mis vastab kokkusurumatule voolamisele rõhukoordinaatide ruumis. Tulemusena saadavas filtreeritud mittelineaarses mudelis (15) säilib energia. Mudel annab parima lähendi filtreerimata dünaamikale lineaarsete protsesside piirjuhul.