Exact Solution of the Linear, 4D-Discrete, Implicit, Semi-Lagrangian Dynamic Equations with Orographic Forcing

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1 Introduction

Exact solution of completely discrete linear equations has multiple outputs:

- Discovery of the exact structure of the solution of a non-stationary numerical model
- Investigation of specific details of flows for optional (non-homogenous) wind and temperature stratification
- Investigation of the influence of vertical discretization (variable Δz) on the quality of numerical models
- Investigation of the time step Δt size impact on the numerical stability and precision of numerical schemes
- Testing of adiabatic cores of NWP models

2 Domain of integration

Geometry is plane: $\overline{p_s}(x, y) \rightarrow p_0 = const.$

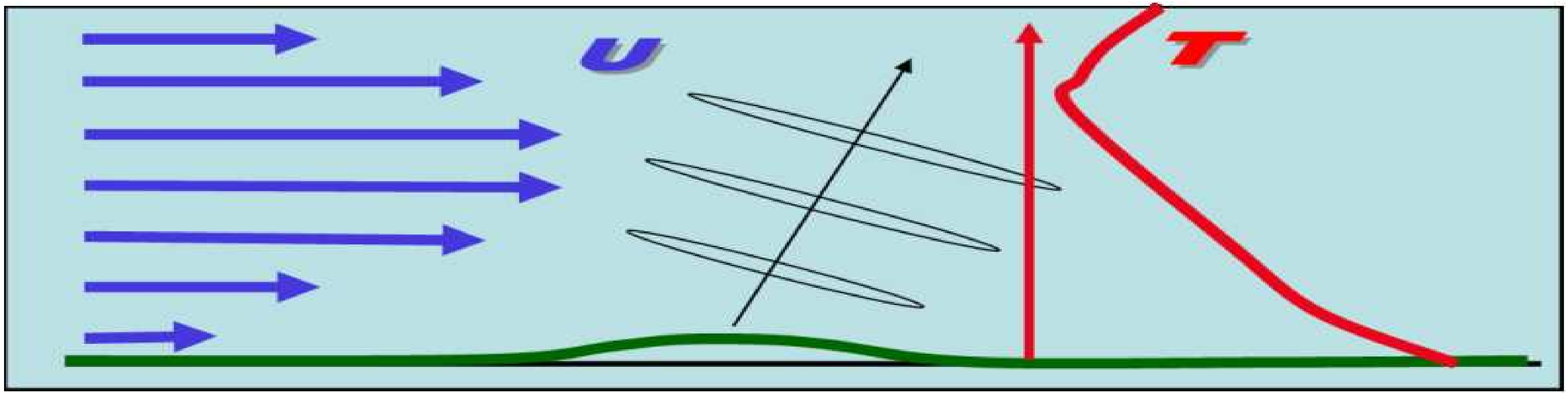
Orography (taken into account in boundary conditions but not in geometry)

is presented by 2D Agnesi profile (h_0 – height, a_x, a_y -half-widths):

$$h(x, y) = \frac{h_0}{1 + (x/a_x)^2 + (y/a_y)^2}.$$

The reference state of the atmosphere is given by wind-speed $U(p)$ and temperature $T(p)$. The Lagrangian time-derivative is approximated as

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + U(p)\frac{\partial}{\partial x}.$$



3 Continuous linear equations

Linear form of White equations (White 1989, Rm 2001). Notation is standard, except of φ which is the nonhydrostatic geopotential fluctuation.

$$
\begin{aligned}
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= -R\frac{p}{H^2}T' - \frac{p^2}{H^2}\frac{\partial\varphi}{\partial p} \ , \\
\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} &= -\nabla\varphi - \mathbf{f}\times\mathbf{u} \ , \\
\frac{\mathrm{d}T'}{\mathrm{d}t} &= \theta\frac{\omega}{p} \ , \\
\nabla\cdot\mathbf{u} + \frac{\partial\omega}{\partial\mathbf{p}} &= \mathbf{0} \ , \\
\frac{\mathrm{d}p_s'}{\mathrm{d}t} + \frac{\mathrm{d}a}{\mathrm{d}t} &= \omega|_{p_0} .
\end{aligned}
\tag{1}
$$

Reference state parameters:

$$H(p) = \frac{RT(p)}{p},$$

$$\theta(p) = \frac{R}{c_p}T(p) - p\frac{\partial T}{\partial p}.$$

Vertical boundary conditions

$$\varphi|_{p_0} = gH(p_0)\frac{p_s'}{p_0} \ .$$

$$\omega|_{p=0} = 0 \ .$$

4 Vertical grid and difference operators

Vertical grid is the Arakawa C-grid. Fields φ_k , $\mathbf{u}_k$, pressure difference Δp_k and non-dimensional height difference $\Delta\zeta_k$ are located on full levels. Pressure $p_{k+1/2}$, $T_{k+1/2}$, $H_{k+1/2}$, $\theta_{k+1/2}$, $\omega_{k+1/2}$ and $T'_{k+1/2}$ belong to the half-levels.

Vertical discretization is originated by $p_{k+1/2}$. Non-dimensional height $\zeta_{k+1/2}$:

$$p_{k+1/2} = p_{klev+1/2}\mathrm{e}^{-\zeta_{k+1/2}}, \quad \zeta_{k+1/2} = \ln(p_{klev+1/2}/p_{k+1/2}).$$

Full-level height and pressure:

$$\zeta_k = \frac{\zeta_{k-1/2} + \zeta_{k+1/2}}{2}, \quad p_k = p_{klev+1/2}\mathrm{e}^{-\zeta_k}.$$

height differences

$$\Delta\zeta_k = \zeta_{k-1/2} - \zeta_{k+1/2}, \quad \Delta\zeta_{k+1/2} = \zeta_k - \zeta_{k+1}.$$

The vertical difference derivatives are

$$\left(\frac{\partial\psi}{\partial p}\right)_k = \frac{\Delta\psi_k}{p_k\Delta\zeta_k}, \quad \left(\frac{\partial\psi}{\partial p}\right)_{k+1/2} = \frac{\Delta\psi_{k+1/2}}{p_{k+1/2}\Delta\zeta_{k+1/2}}.$$

5 Horizontal grid and difference operators

$$\psi^n_{ijk} = \psi(x_i, y_j, p_k, t_n), \quad \psi^{n-1}_{i-l_k,jk} = \psi(x_i - U_k\Delta t, y_j, p_k, t_{n-1}),$$

$l_k = \dfrac{U_k\Delta t}{\Delta x}$, – non-dimensional displacement along x-axes during Δt.

$$\delta_t\psi^n_{ijk} = \frac{\psi^n_{ij,k} - \psi^{n-1}_{i-l_k,j,k}}{\Delta t}, \quad \overline{\psi}^n_{ijk} = \frac{\psi^n_{ijk} + \psi^{n-1}_{i-l_k,jk}}{2}$$

$$\delta_i\psi = \frac{1}{\Delta x}\left(\psi_{i+1/2} - \psi_{i-1/2}\right), \quad \delta_{i+1/2}\psi = \frac{1}{\Delta x}\left(\psi_{i+1} - \psi_i\right),$$

$$\delta_j\psi = \frac{1}{\Delta y}\left(\psi_{j+1/2} - \psi_{j-1/2}\right), \quad \delta_{j+1/2}\psi = \frac{1}{\Delta y}\left(\psi_{j+1} - \psi_j\right)$$

$$\overline{\psi}^x_i = \frac{1}{2}\left(\psi_{i-1/2} + \psi_{i+1/2}\right), \quad \overline{\psi}^x_{i+1/2} = \frac{1}{2}\left(\psi_i + \psi_{i+1}\right),$$

$$\overline{\psi}^y_j = \frac{1}{2}\left(\psi_{j-1/2} + \psi_{j+1/2}\right), \quad \overline{\psi}^y_{j+1/2} = \frac{1}{2}\left(\psi_j + \psi_{j+1}\right)$$

6 Semi-Implicit, Semi-Lagrangian Equations

Equations (1) become in SISL presentation

$$\delta_t \omega^n_{ijk+1/2} = -R\left(\frac{p}{H^2}\right)_{k+1/2} \overline{T'}^n_{ijk+1/2} - \left(\frac{p}{H^2 \Delta\zeta}\right)_{k+1/2} \overline{\Delta\varphi}^n_{ijk+1/2} ,$$

$$\delta_t u^n_{i+1/2,jk} = -\overline{(\delta_x \varphi)}^n_{i+1/2,jk} + f\overline{\overline{v}^{xy}}^n_{i+1/2,jk} ,$$

$$\delta_t v^n_{ij+1/2,k} = -\overline{(\delta_y \varphi)}^n_{ij+1/2,k} - f\overline{\overline{u}^{xy}}^n_{ij+1/2,k} ,$$

$$\delta_t {T'}^n_{ijk+1/2} = \left(\frac{\theta}{p}\right)_{k+1/2} \overline{\omega}^n_{ijk+1/2} , \tag{2}$$

$$(\nabla \cdot \mathbf{u})^n_{ijk} + \frac{\Delta\omega^n_{ijk}}{(p\Delta\zeta)_k} = 0 ,$$

$$\delta_t(p_s')^n_{ij,klev+1/2} + \delta_t(a)_{ij,klev+1/2} = \overline{\omega}^n_{ijklev+1/2}.$$

7 Embedding into 3D Fourier space

In following, the discrete Fourier transformation is employed in the form

$$f_i = \sum_{q=0}^{2N-1} \hat{f}_q e^{\mathbf{i}\eta_q i}, \text{ kus } \hat{f}_q = \frac{1}{2N}\sum_{i=0}^{2N-1} f_q e^{-\mathbf{i}\eta_q i} \text{ ja } \eta_q = \pi\frac{q}{N} .$$

Orography in the 2D Fourier basis is

$$a_{ij} \equiv a(x_i, y_j) = \sum_{qr} \hat{a}_{qr} e^{\mathbf{i}(\eta^x_q i + \eta^y_r j)},$$

Using short notation $\psi^n_{ijk} = \left\{T'^n_{ijk+1/2}, \omega^n_{ijk+1/2}, u^n_{i+1/2jk}, v^n_{ij+1/2k}, \varphi^n_{ijk}, (p_s')^n_{ij}\right\}$

A general presentation holds

$$\psi^n_{ijk} = \sum_{qrs} \hat{\psi}^s_{qrk} e^{\mathbf{i}(\eta^x_q i + \eta^y_r j - d^s_{qrk} n)}, \tag{3}$$

where d^s_{qrk} are the non-dimensional frequencies.

8 Normal mode equations: (3) $\to$ (2) :

$$\mathbf{i}\nu^s_{qrk+1/2}\hat{\omega}^s_{qr,k+1/2} = -R\left(\frac{p}{H^2}\right)_{k+1/2}\hat{T}^s_{qrk+1/2} - \left(\frac{p}{H^2\Delta\zeta}\right)_{k+1/2}\Delta\hat{\varphi}^s_{qrk+1/2}\,,$$

$$\mathbf{i}\nu^s_{qrk}\hat{u}^s_{qrk} = -\mathbf{i}\mu^x_q\hat{\varphi}^s_{qrk} + F_{qr}\hat{v}^s_{qrk}, \quad \mathbf{i}\nu^s_{qrk}\hat{v}^s_{qrk} = -\mathbf{i}\mu^y_r\hat{\varphi}^s_{qrk} - F^*_{qr}\hat{u}^s_{qrk}, \quad (4)$$

$$\mathbf{i}\nu^s_{qrk+1/2}\hat{T}^s_{qrk+1/2} = \frac{\theta_{k+1/2}}{p_{k+1/2}}\hat{\omega}^s_{qrk+1/2}, \quad \mathbf{i}[(\mu^x_q)^*\hat{u}^s_{qrk} + (\mu^y_r)^*\hat{v}^s_{qrk}] + \frac{\Delta\hat{\omega}^s_{qrk}}{(p\Delta\zeta)_k} = 0\,.$$

$$F_{qr} = f a^x_q (a^y_r)^*, \; a^x_q = \frac{1+\mathrm{e}^{\mathbf{i}\eta^x_q}}{2}, \; a^y_r = \frac{1+\mathrm{e}^{\mathbf{i}\eta^y_r}}{2}, \; \mu^x_q = \frac{\mathrm{e}^{\mathbf{i}\eta^x_q}-1}{\mathbf{i}\Delta x}, \; \mu^y_r = \frac{\mathrm{e}^{\mathbf{i}\eta^y_r}-1}{\mathbf{i}\Delta y}.$$

$$\nu^s_{qrk} = \frac{2}{\Delta t}\tan[\frac{1}{2}(\eta^x_q l_k - d^s_{qrk})].$$

Stationary case $s=0$: $d^0_{qrk} = 0 \;\Rightarrow\; \nu^s_{qrk} = \nu^0_{qk} = \frac{2}{\Delta t}\tan(\frac{1}{2}\eta^x_q l_k)$.

Non-stationary case $s \neq 0$: $d^s_{qrk} = \eta^x_q l_k - 2\tilde{d}^s_{qr} \;\Rightarrow\; \nu^s_{qrk} = \nu^s_{qr} = \frac{2}{\Delta t}\tan(d^s_{qr})$.

9 Wave equation for vertical velocity $\hat{\omega}^s_{qrk+1/2}$

From normal mode equations, the wave equation follows for $\hat{\omega}^s_{qrk+1/2}$

$$(\mathcal{L}\hat{\omega}^s)_{qrk+1/2} + \lambda^s_{qrk+1/2}\hat{\omega}^s_{qrk+1/2} = 0, \tag{5}$$

where

$$\lambda^s_{qrk+1/2} = H^2_{k+1/2}\mu^2_{qr}\frac{(\nu^s_{qrk+1/2})^2 - N^2_{k+1/2}}{|F_{qr}|^2 - (\nu^s_{qrk+1/2})^2}.$$

$$(\mathcal{L}\hat{\omega}^s)_{qrk+1/2} = L^+_{qrk+1/2}\Delta\hat{\omega}^s_{qrk+1} - L^-_{qrk+1/2}\Delta\hat{\omega}^s_{qrk},$$

$$L^+_{qrk+1/2} = \frac{\tilde{\nu}^s_{qrk+1}}{\tilde{\nu}^s_{qrk+1/2}}\frac{\mathrm{e}^{-\Delta\zeta_{k+1}/2}}{\Delta\zeta_{k+1/2}\Delta\zeta_{k+1}}, \quad L^-_{qrk+1/2} = \frac{\tilde{\nu}^s_{qrk}}{\tilde{\nu}^s_{qrk+1/2}}\frac{\mathrm{e}^{\Delta\zeta_k/2}}{\Delta\zeta_{k+1/2}\Delta\zeta_k},$$

$$\tilde{\nu}^s_{qrk} = \frac{(\nu^s_{qrk})^2 - |F_{qr}|^2}{\nu^s_{qrk}}.$$

10 Stationary solution

10.1 Main recurrence formula

Solutions of Eq. (5) are sought in the form

$$\chi_{k+1/2} = \chi_{1/2} \prod_{j=0}^{k} c_j, \qquad \chi_{1/2} = 1. \tag{8}$$

From (5), a nonlinear, two-point recurrence follows for growth factors c_k

$$L^{+}_{k+1/2}(c_{k+1} - 1) + L^{-}_{k+1/2}(1/c_k - 1) + \lambda_{k+1/2} = 0. \tag{9}$$

The initial value $c(0)$, required to start this simple straightforward formula, can be established, assuming the homogeneous layer in the top in the free buoyancy-wave radiating state. The solution then presents

$$\omega^0_{qrk+1/2} = C\chi_{k+1/2},$$

where C is specified from boundary condition on the surface.

10.2 Special case of homogeneous stratification

Homogeneous atmosphere : $\Delta\zeta_k = \Delta\zeta_{k+1/2} = \Delta\zeta, \quad L^{\pm}_{k+1/2} = \dfrac{e^{\mp\Delta\zeta/2}}{(\Delta\zeta)^2},$

Homogeneous atmosphere : $\lambda^0_{qrk} = \lambda^0_{qr} = H^2\mu^2_{qr}\dfrac{(\nu^0_{qr})^2 - N^2}{|F_{qr}|^2 - (\nu^0_{qr})^2}.$

Consequently $\quad c_k = c, \qquad \chi_{k+1/2} = \chi_{1/2}c^k,$

where c satisfies equation $\quad e^{-\Delta\zeta/2}(c-1) + e^{\Delta\zeta/2}(c^{-1}-1) + (\Delta\zeta)^2\lambda = 0.$

The solution of this equation is:

trapped wave, $\quad Q \ \geq\ 1: \quad c = e^{\Delta\zeta/2}\left(Q + \sqrt{Q^2-1}\right),$

free wave, $\quad |Q| \ <\ 1: \quad c = e^{(\Delta\zeta/2 - \mathbf{i}\Delta\xi)}, \quad \Delta\xi = \mathrm{Arctan}\dfrac{\sqrt{1-Q^2}}{Q},$

where

$$Q \equiv Q_{qr} = \cosh(\Delta\zeta/2) - \frac{1}{2}(\Delta\zeta)^2\lambda^0_{qr}.$$

11 Non-stationary free waves

Assuming

$$\nu^s_{qrk+1/2} = \nu^s_{qr},$$

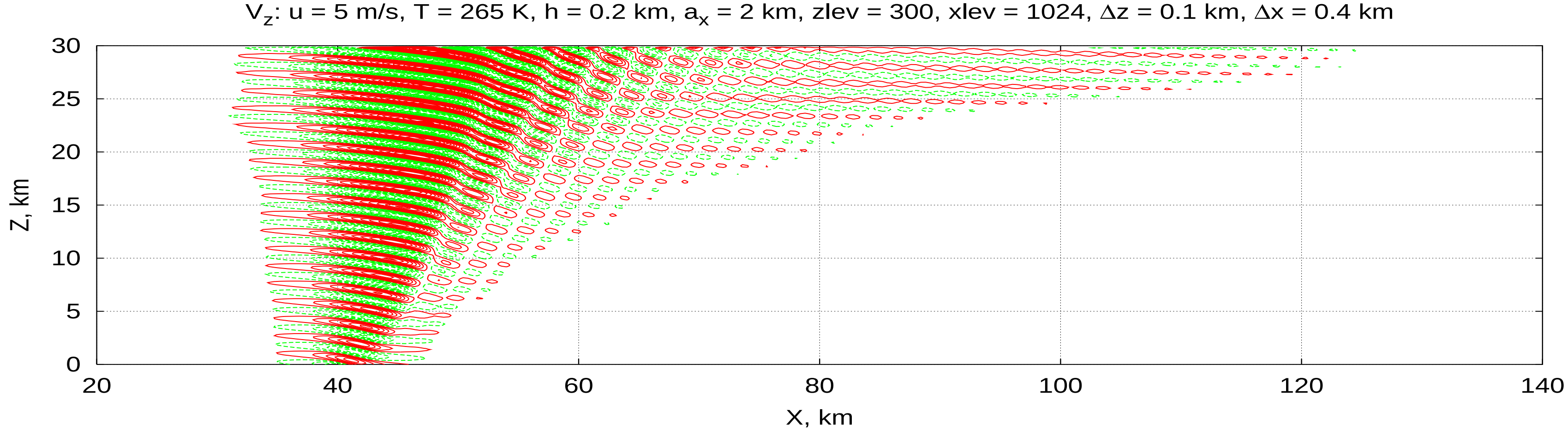
the Eq. (5) simplifies, as the coefficients $L^{\pm}$ become independent of the frequency (and hence, of horizontal wave-numbers q, r):

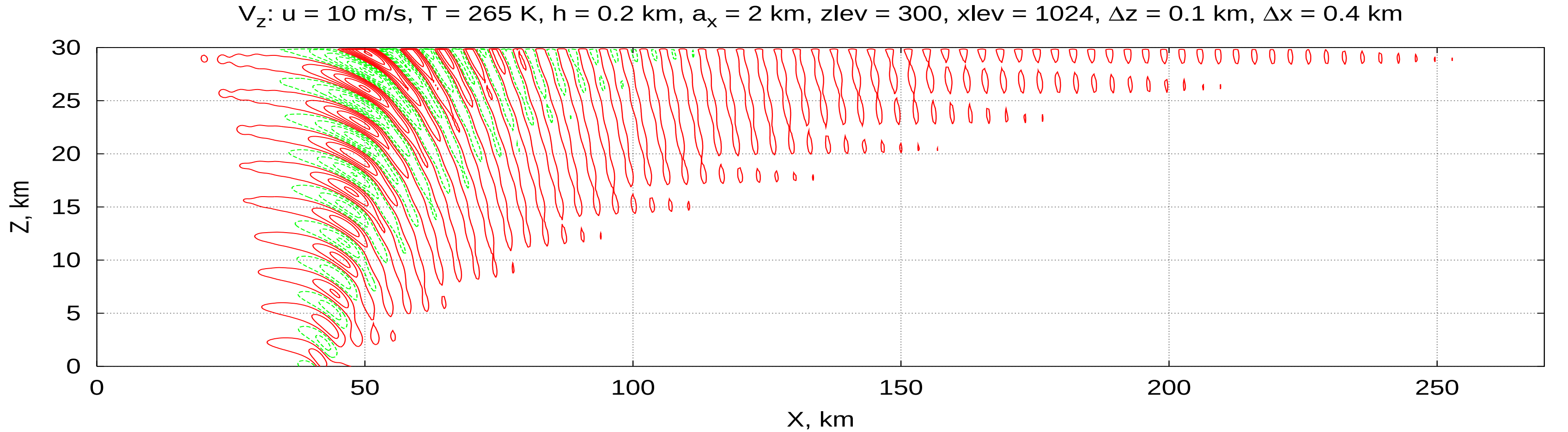
$$L^+_{k+1/2} = \frac{p_{k+1/2}/p_{k+1}}{\Delta\zeta_{k+1/2}\Delta\zeta_{k+1}} = \frac{\mathrm{e}^{-\Delta\zeta_{k+1}/2}}{\Delta\zeta_{k+1/2}\Delta\zeta_{k+1}},$$

$$L^-_{k+1/2} = \frac{p_{k+1/2}/p_k}{\Delta\zeta_{k+1/2}\Delta\zeta_k} = \frac{\mathrm{e}^{\Delta\zeta_k/2}}{\Delta\zeta_{k+1/2}\Delta\zeta_k}.$$

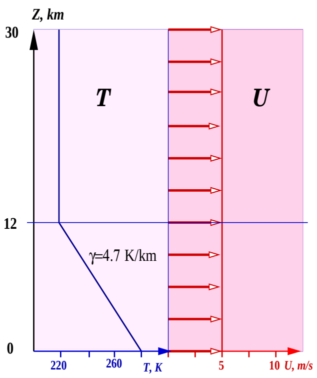
Quest quantities are the frequencies ν^s_{qr} in λ and the corresponding eigenfunctions. The solution presents as a combination of functions (8), found from recurrence (9). The wanted frequencies are selected by the homogeneous bottom boundary condition.

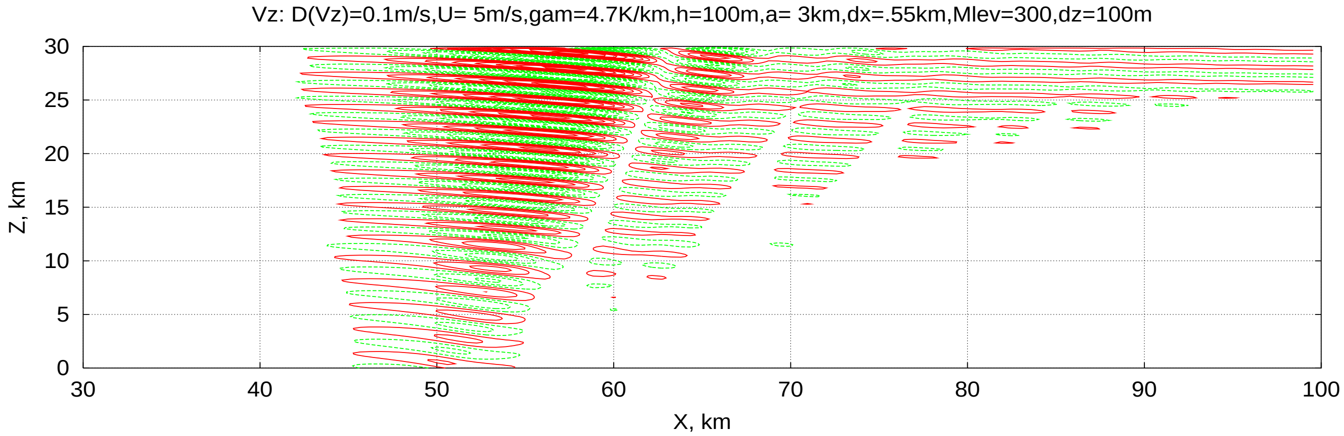
12 Examples: homogeneous stratification

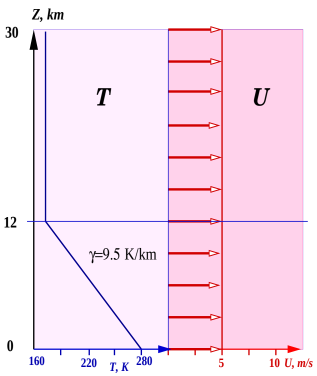


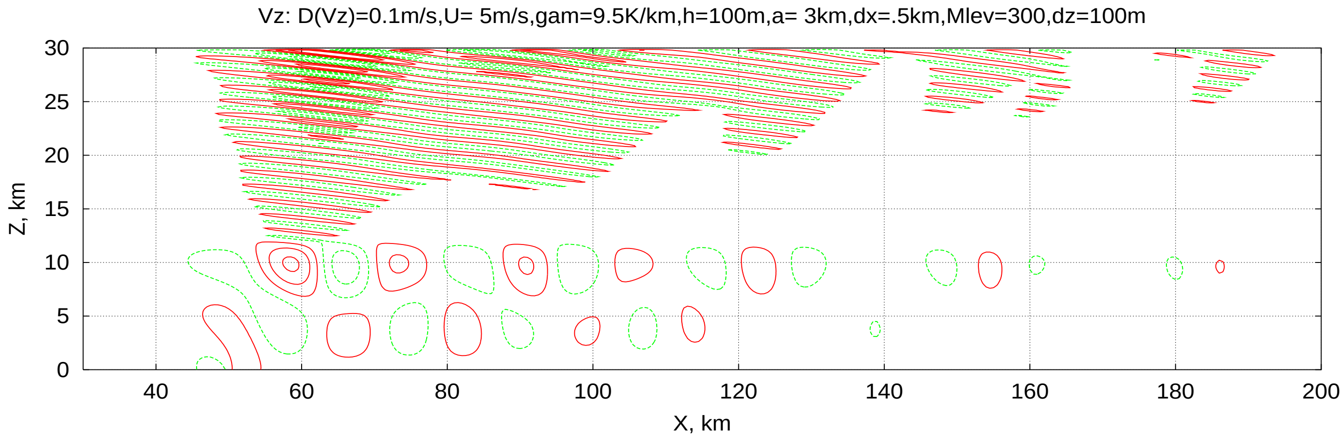


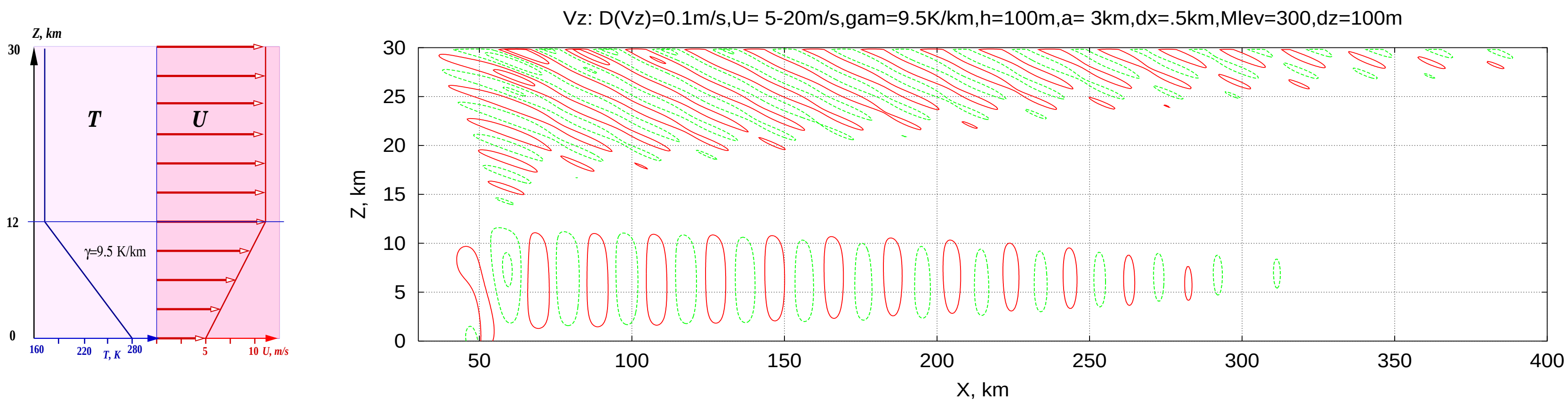
13 Examples: BW refraction on tropopause










Z, km
30
12
0
T
U
γ=9.5 K/km
160
220
T, K
280
5
10 U, m/s
Vz: D(Vz)=0.1m/s,U= 5-20m/s,gam=9.5K/km,h=100m,a= 3km,dx=.5km,Mlev=300,dz=100m
Z, km
X, km
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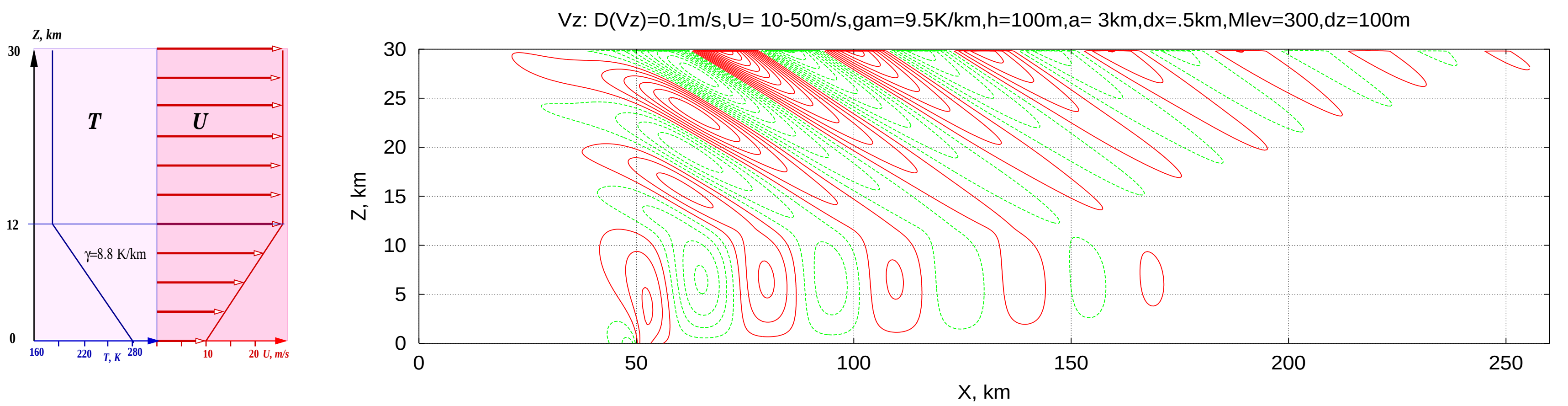
Z, km
30
12
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T
U
γ=8.8 K/km
160
220
T, K
280
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20 U, m/s
Vz: D(Vz)=0.1m/s,U= 10-50m/s,gam=9.5K/km,h=100m,a= 3km,dx=.5km,Mlev=300,dz=100m
Z, km
X, km
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100
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200
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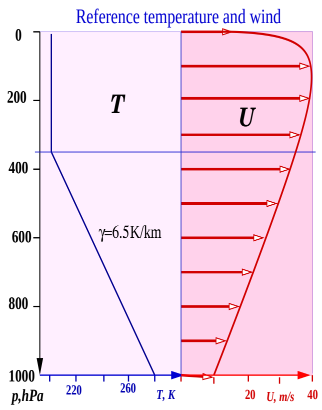
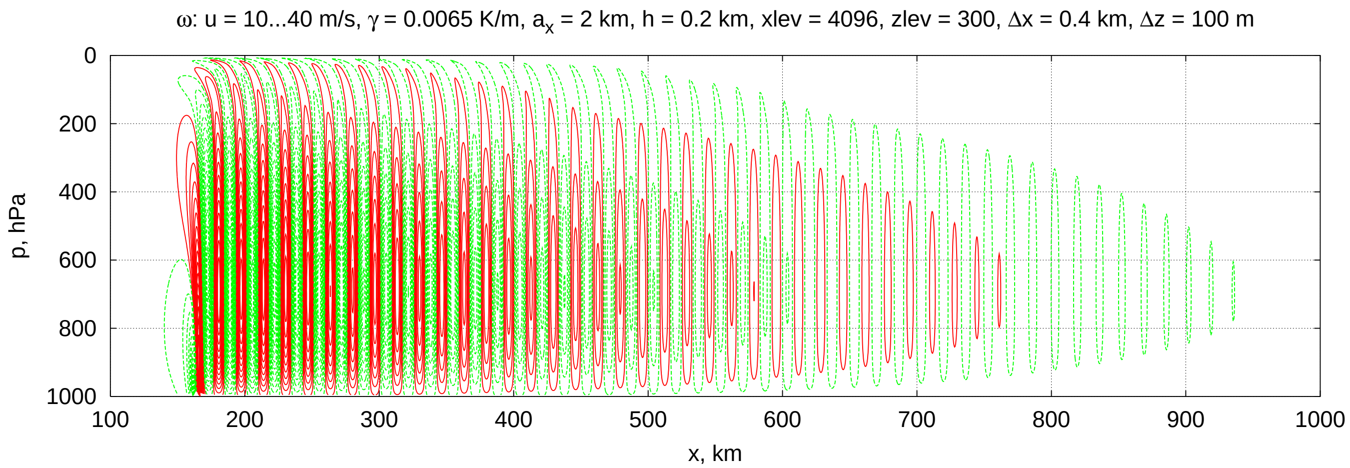
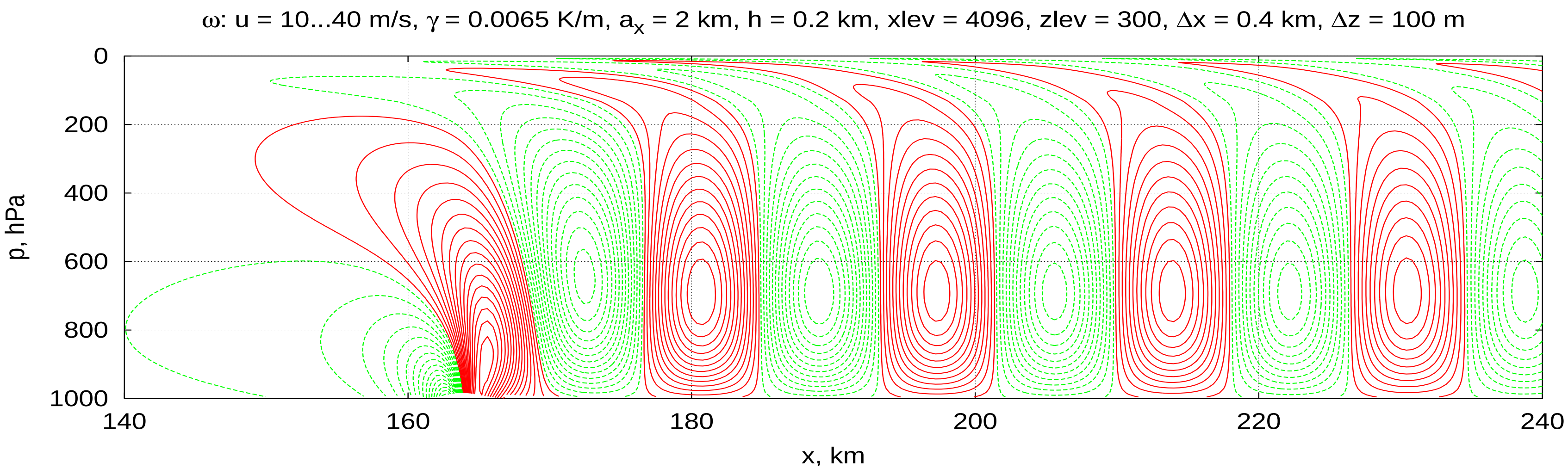
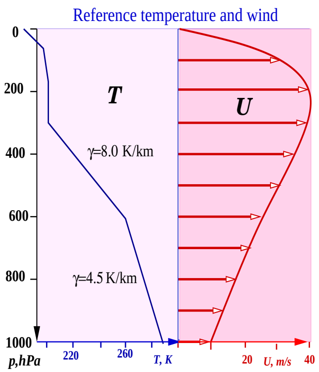
14 Examples: Hyperbolic reference wind profile

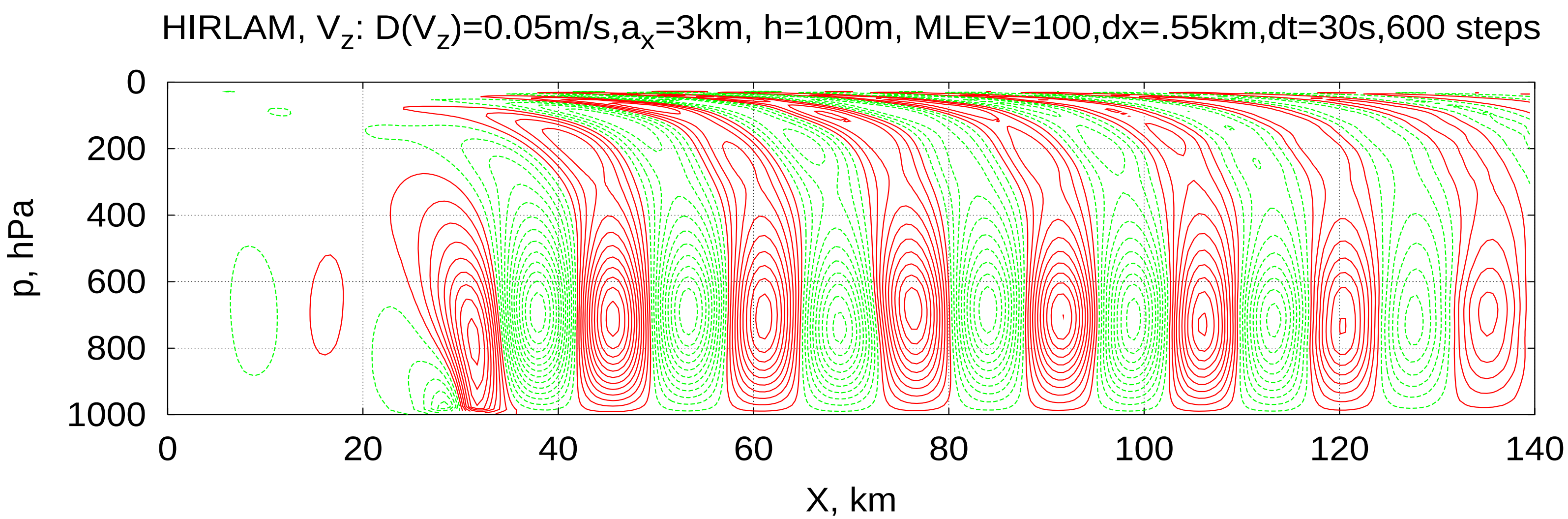


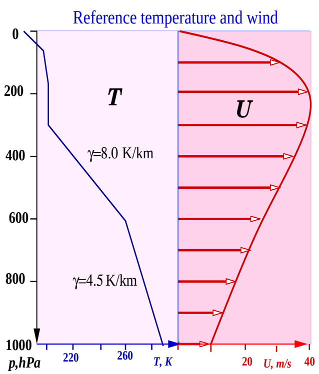


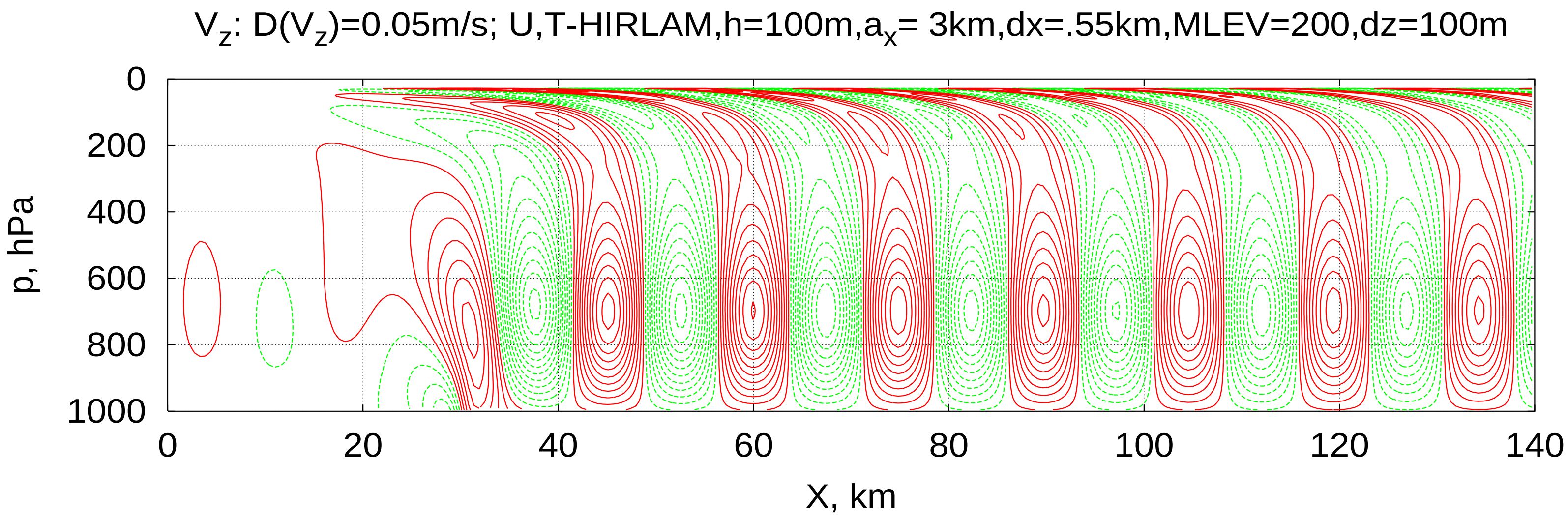


15 Example: Testing NH HIRLAM









16 References

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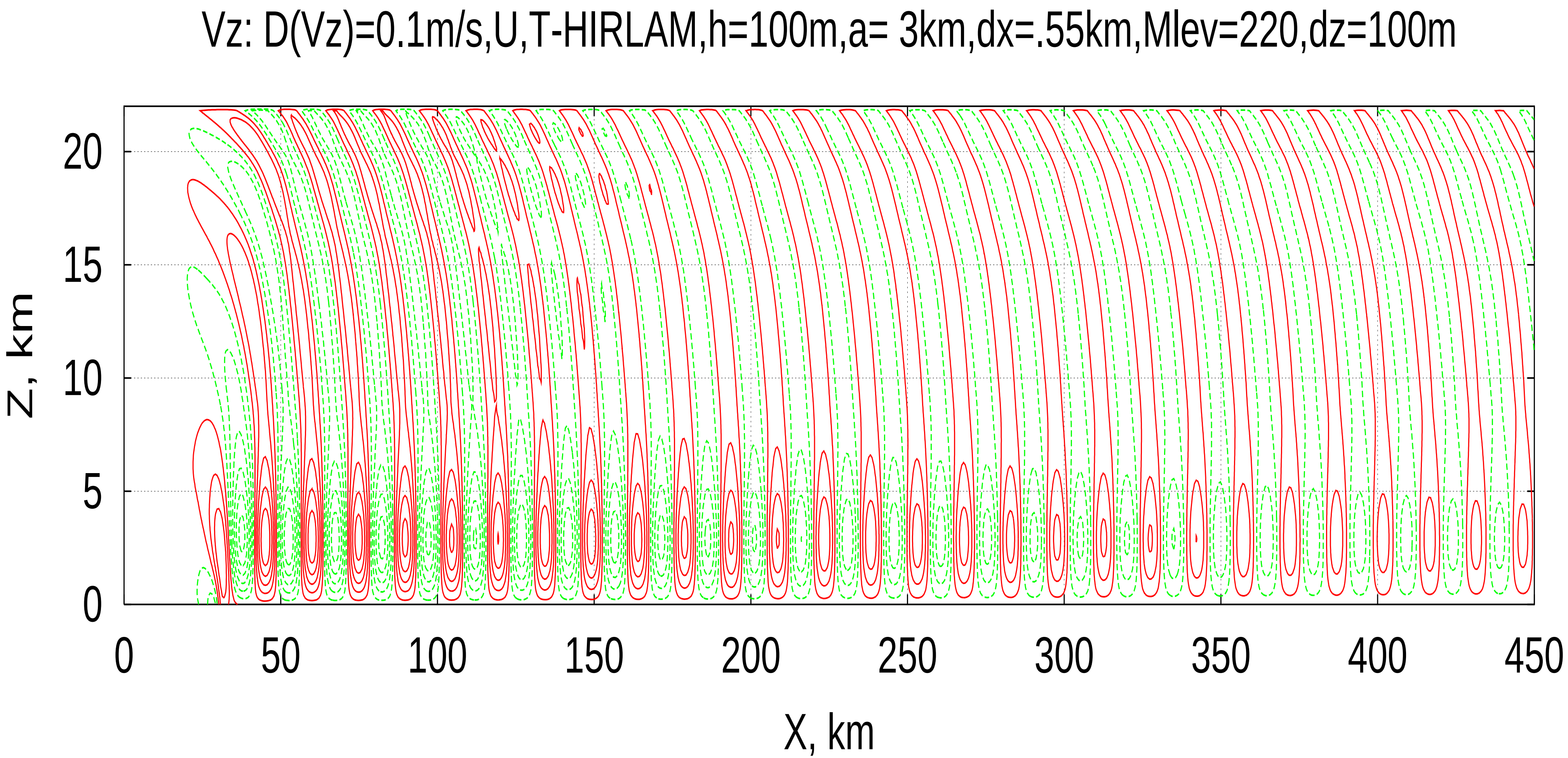
Rõõm, R., 2001: Nonhydrostatic adiabatic kernel for HIRLAM. Part I: Fundametals of nonhydrostatic dynamics in pressure-related coordinates. *HIRLAM Technical Report*, **48**, 26p.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z, km
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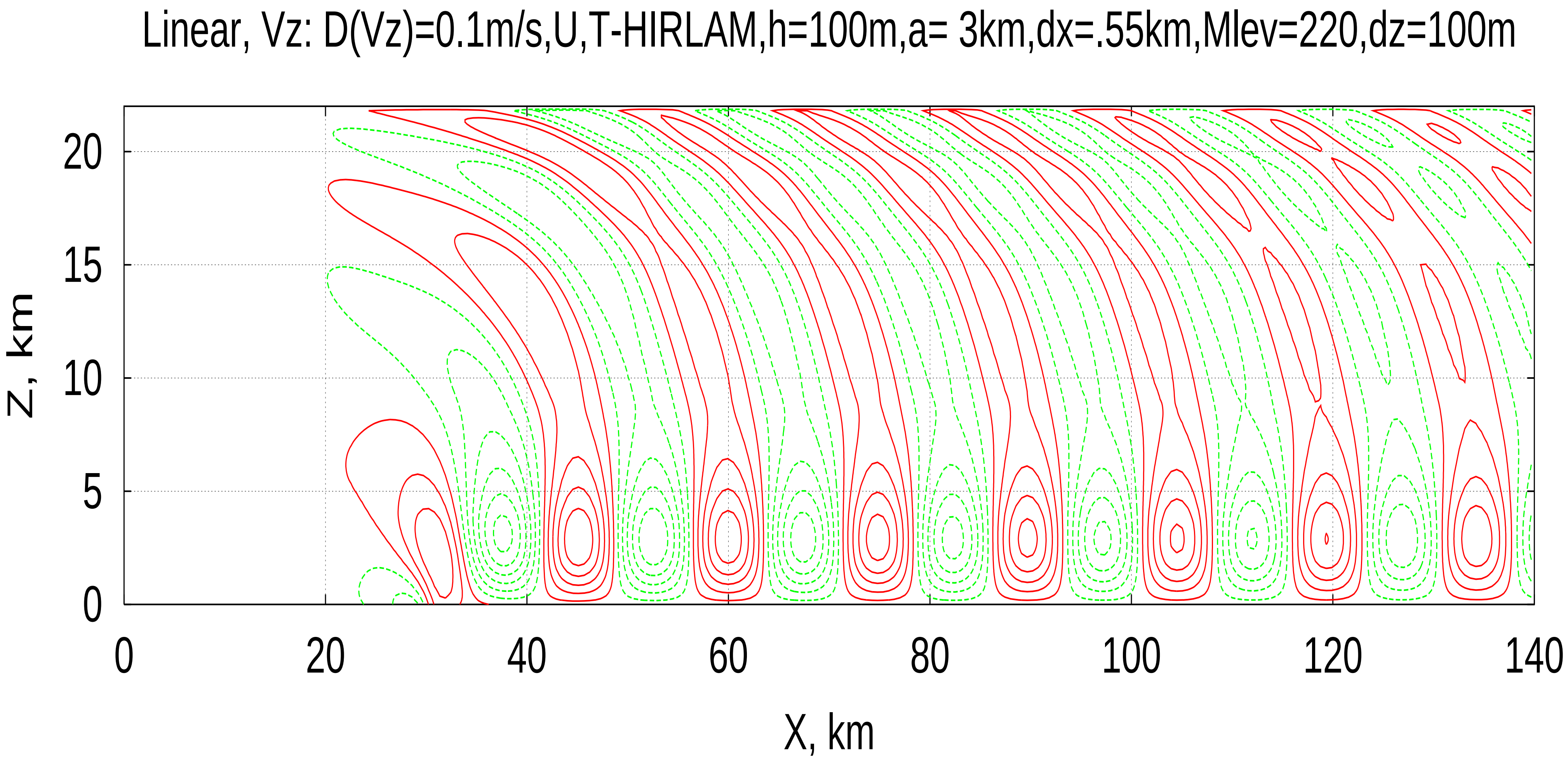
X, km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z, km
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80
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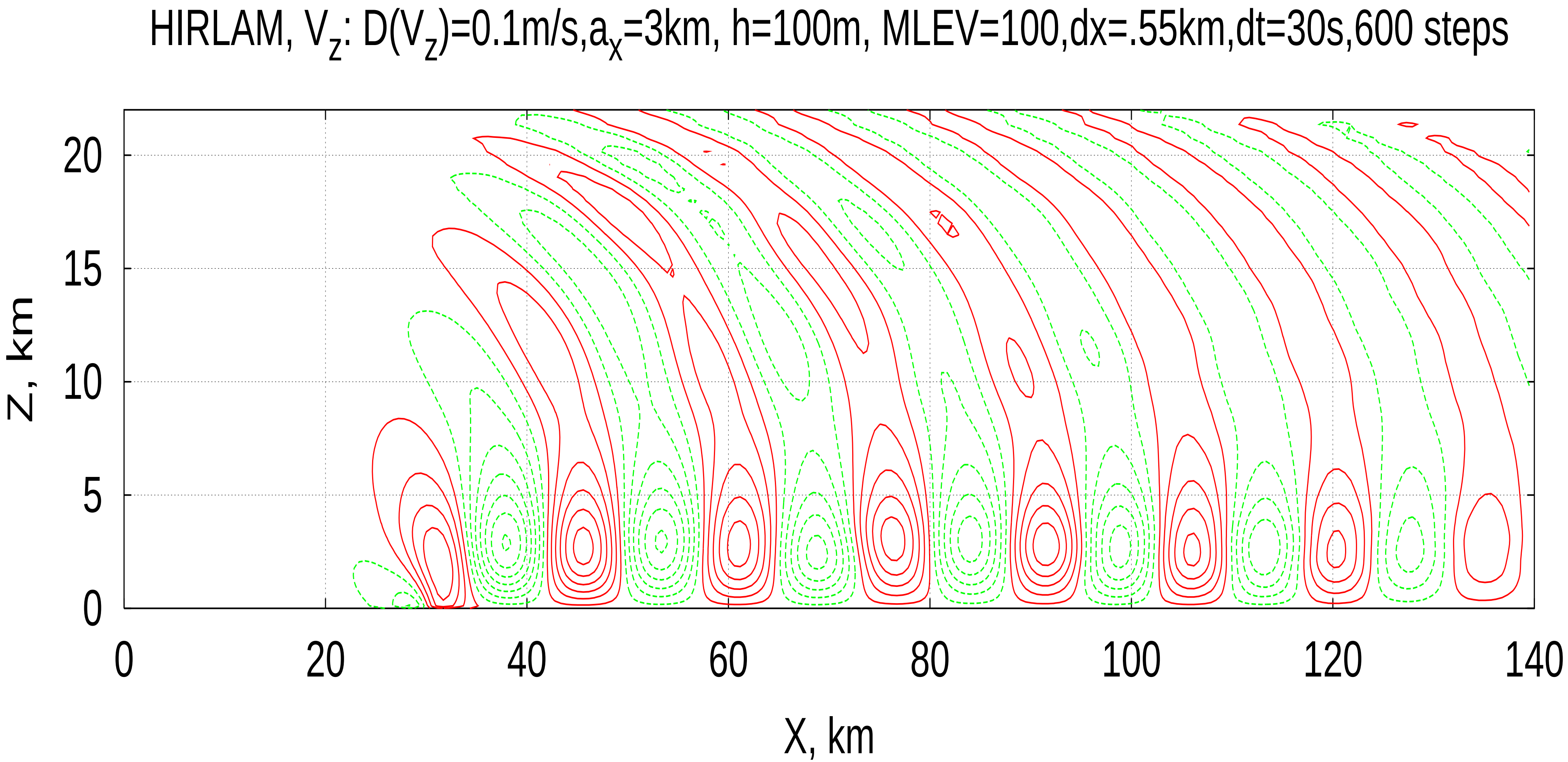
120
140
X, km



HIRLAM, V_z: $D(V_z)$=0.1m/s,a_x=3km, h=100m, MLEV=100,dx=.55km,dt=30s,600 steps
z, km
0
5
10
15
20
0
20
40
60
80
100
120
140
X, km