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Radiation field in a semi-infinite homogeneous atmosphere with internal sources

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Abstract

The equation of radiative transfer in a semi-infinite homogeneous atmosphere with different internal sources is solved by the method of kernel approximation—the kernel in the equation for the Sobolev resolvent function is approximated by a Gauss–Legendre sum. Then the obtained approximate equation can be solved exactly and the solution is a weighted sum of exponentials. All the necessary coefficients of the solutions may be easily found. Since the resolvent function is closely connected with the Green function of the integral radiative transfer equation, the radiation field for different internal sources can be found by simple integration. For the considered cases the formulas for the radiation field are obtained and the respective accuracy estimated. The package of codes in Fortran-77 is given at http://www.aai.ee/~viik/homogen.for.

Keywords: Radiative transfer; Kernel approximation

1. Introduction

We are interested in determining the radiation field in a semi-infinite homogeneous isotropically scattering atmosphere with different internal sources. This same problem with exponential sources and the Milne problem have been studied in hundreds of books and papers—here we note only some of them [1–10]. The methods to solve the equation of radiative transfer in everyday use in astrophysics, planetary physics and neutron transfer are much more complicated and suited for finding the radiation field not only in highly simplified models but in real systems—stellar and planetary atmospheres and in nuclear reactors. However, there is still a need for simple and reliable benchmark methods which is what this paper tries to present. Our approach is based on the kernel approximation method first proposed by Krook [10] and later developed by Gybicki [12]. Viik et al. [13] published a length paper together with Fortran codes but in not so well-known Tartu Observatory proceedings and in Russian.

Here, we first describe the theoretical background which has been best elaborated by Sobolev and his colleagues in then Leningrad, now St. Petersburg, University, e.g. [3,7].

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We give a systematic treatment of the cases with different internal sources, producing formulas for the source functions and the intensities of radiation called forth by the same internal sources. In order to simplify the respective formulas we introduce $h(\tau,\mu)$ and $g(\tau,\mu)$ functions where one of them, the h function is a generalization of the well-known Ambarzumian–Chandrasekhar H function. We give both the integral and differential equations which these two functions satisfy.

The respective Fortran codes can be found at http://www.aai.ee/~viik/homogen.for. This package contains also the codes for the optically finite homogeneous atmospheres.

2. Theoretical background

Let us now consider the determination of radiation field in an isotropically scattering homogeneous optically semi-infinite atmosphere. In this case the source function B is described by the integral equation

$$B(\tau)=\frac{1}{2}\lambda\int_0^\infty E_1(t-\tau)B(t)\,\mathrm{d}t+B_0(\tau), \tag{1}$$

where

$$E_n(x)=\int_0^1 \exp(-|x|s)s^{n-2}\,\mathrm{d}s$$

and $B_0(\tau)$ describes the distribution of the internal sources. We define the resolvent of Eq. (1) as follows:

$$\Gamma(\tau,\tau')=\frac{1}{2}\lambda\int_0^\infty E_1(t-\tau)\Gamma(t,\tau')\,\mathrm{d}t+\frac{1}{2}\lambda E_1(\tau-\tau'). \tag{2}$$

The solution of Eq. (1) can be expressed as

$$B(\tau)=B_0(\tau)+\int_0^\infty \Gamma(\tau,t)B_0(t)\,\mathrm{d}t. \tag{3}$$

It is easy to see that the resolvent is simply the regular part of the Green function for Eq. (1). Sobolev proved that the resolvent $\Gamma(\tau,\tau')$ can be expressed in terms of a function with lesser number of arguments $\Phi(\tau)$ [3], namely

$$\Phi(\tau)=\Gamma(0,\tau)=\Gamma(\tau,0),$$

while the symmetry of the resolvent follows from Eq. (2), and

$$\frac{\partial\Gamma(\tau,\tau')}{\partial\tau}+\frac{\partial\Gamma(\tau,\tau')}{\partial\tau'}=\Phi(\tau)\Phi(\tau'). \tag{4}$$

Considering Eq. (4) we have

$$\Gamma(\tau,\tau')=\Phi(|\tau-\tau'|)+\int_0^{\bar{\tau}}\Phi(\tau-t)\Phi(\tau'-t)\,\mathrm{d}t, \tag{5}$$

where $\bar{\tau}=\min(\tau,\tau')$.

The function $\Phi(\tau)$ can be found from the following equation:

$$\Phi(\tau)=\frac{1}{2}\lambda\int_0^\infty E_1(t-\tau)\Phi(t)\,\mathrm{d}t+\frac{1}{2}\lambda E_1(\tau). \tag{6}$$

Next we approximate the kernel of Eq. (6) by a sum of exponentials:

$$E_1(\tau)=\sum_{n=1}^{N} w_n\exp(-\tau u_n)u_n^{-1},$$

where w_n are the weights and u_n are the points of the Gauss quadrature rule of the order N in the interval (0,1). After having substituted this approximation into Eq. (6) it can be solved exactly and the solution is

$$\Phi(\tau) = \sum_{i=1}^{N} a_i \exp(-s_i\tau), \tag{7}$$

where the coefficients s_i can be found from the characteristic equation

$$1 - \lambda \sum_{n=1}^{N} \frac{w_n}{1 - s^2 u_i^2} = 0. \tag{8}$$

The coefficients a_i are to be found from a linear algebraic system of equations

$$\sum_{i=1}^{N} \frac{a_i}{1 - s_i i u_j} = u_j^{-1}, \quad j = 1, \ldots, N.$$

Characteristic equation (8) is a polynomial of the order N with respect to s^2. Consequently, it has N pairs of zeros $\pm s_k$ which satisfy the inequalities

$$0 \leqslant s_1 < u_N^{-1} < s_2 < u_{N-1}^{-1} < s_3 < \cdots < s_N < u_1^{-1}.$$

It is evident that if we deal with the conservative atmosphere, i.e. $\lambda = 1$, then $s = \pm 0$ is the zero too.

Next we define two new functions which come handy in the following:

$$h(\tau,\mu) = 1 + \int_{\tau}^{\infty} \Phi(t) \mathrm{e}^{-(t-\tau)/\mu}\, \mathrm{d}t \tag{9}$$

and

$$g(\tau,\mu) = \mathrm{e}^{-\tau/\mu} + \int_{0}^{\tau} \Phi(t) \mathrm{e}^{-(\tau-t)/\mu}\, \mathrm{d}t. \tag{10}$$

Here and in all the following formulas it is assumed that $\mu \geqslant 0$.

The function $h(\tau,\mu)$ is a generalization of the well-known Ambarzumian–Chandrasekhar function $H(\mu)$ since

$$H(\mu) = 1 + \int_{0}^{\infty} \Phi(t) \mathrm{e}^{-t/\mu}\, \mathrm{d}t \tag{11}$$

and therefore

$$h(0,\mu) = H(\mu).$$

Using Eq. (7) in Eqs. (9)–(11) we find that

$$h(\tau,\mu) = 1 + \mu \sum_{i=1}^{N} \frac{a_i \mathrm{e}^{-s_i\tau}}{1 + s_i\mu}, \tag{12}$$

$$H(\mu) = 1 + \mu \sum_{i=1}^{N} \frac{a_i}{1 + s_i\mu}, \tag{13}$$

$$g(\tau,\mu) = \mathrm{e}^{-\tau/\mu} + \mu \sum_{i=1}^{N} \frac{a_i(\mathrm{e}^{-s_i\tau} - \mathrm{e}^{-\tau/\mu})}{1 - s_i\mu}. \tag{14}$$

There is no singularity in Eq. (14) since if $s_i\mu = 1$ the respective term in the sum is $a_i\tau\mu^{-1}$. From Eqs. (9), (10), (12) and (14) it follows that

$$g(0,\mu) = 1, \tag{15}$$

$$g(\tau,\infty) = 1 + \sum_{i=1}^{N} a_i s_i^{-1}(1 - \mathrm{e}^{-s_i\tau}), \tag{16}$$

$$h(\infty,\mu) = 1, \tag{17}$$

$$h(\tau,\infty) = 1 + \sum_{i=1}^{N} a_i s_i^{-1}\mathrm{e}^{-s_i\tau}. \tag{18}$$

For the conservative case ($\lambda = 1$) we cannot use Eqs. (16) and (18) since then the first zero of the characteristic equation (8) $s_1 = 0$.

3. Equations for *h* and *g*

By differentiating Eqs. (9) and (10) with respect to τ we obtain differential equations for functions h and g:

$$-\mu\frac{\partial h(\tau,\mu)}{\partial\tau} + h(\tau,\mu) = \mu\Phi(\tau) + 1, \tag{19}$$

$$\mu\frac{\partial g(\tau,\mu)}{\partial\tau} + g(\tau,\mu) = \mu\Phi(\tau). \tag{20}$$

If we now define function $b(\tau,\pm\mu)$ by expressions

$$\mu b(\tau,-\mu) = h(\tau,\mu) - 1,$$

$$\mu b(\tau,\mu) = g(\tau,\mu),$$

then we have

$$-\mu\frac{\partial b(\tau,-\mu)}{\partial\tau} + b(\tau,-\mu) = \Phi(\tau), \tag{21}$$

$$\mu\frac{\partial b(\tau,\mu)}{\partial\tau} + b(\tau,\mu) = \Phi(\tau). \tag{22}$$

It is evident that our h and g functions are closely connected with the b function, defined by Kagiwada et al. [8]. From Eqs. (21) and (22) it follows that function b is the intensity in a semi-infinite homogeneous atmosphere which is illuminated according to the law

$$b(0,\mu) = \mu^{-1}.$$

For such a problem the source function is

$$\Phi(\tau) = \frac{1}{2}\lambda\int_{-1}^{+1} b(\tau,\mu')\,\mathrm{d}\mu',$$

or, taking into account Eqs. (21) and (22), we have

$$\Phi(\tau) = \frac{1}{2}\lambda\int_{0}^{1} [h(\tau,\mu') + g(\tau,\mu') - 1]\,\mathrm{d}\mu'/\mu'.$$

We may obtain integral equations for functions h and g. If the homogeneous semi-infinite atmosphere is illuminated by a parallel beam which produces the flux $\pi F\mu_0$ at the boundary of the atmosphere we have from Eqs. (1) and (6) that

$$\Phi(\tau) = 2F^{-1}\int_{0}^{1} B(\tau,\mu')\,\mathrm{d}\mu'/\mu', \tag{23}$$

where $B(\tau,\mu_0)$ is the source function for such an atmosphere. Substituting Eq. (23) into Eq. (9) we have

$$h(\tau,\mu) = 1 + 2\mu F^{-1}\int_{0}^{1} I(\tau,-\mu,\mu')\,\mathrm{d}\mu'/\mu', \tag{24}$$

where $I(\tau,-\mu,\mu')$ is the intensity in an externally illuminated atmosphere, cf. Eq. (32). From Eq. (34) we have

$$I(\tau,-\mu,\mu') = \frac{1}{4}\lambda F \frac{\mu' H(\mu')}{\mu+\mu'}[h(\tau,\mu)+g(\tau,\mu')-1],$$

and as a result we obtain from Eq. (24) that

$$h(\tau,\mu) = 1+\frac{1}{2}\lambda\mu H(\mu)\int_0^1 \frac{H(\mu')g(\tau,\mu')}{\mu+\mu'}\,\mathrm{d}\mu'. \tag{25}$$

Here, we have used the Chandrasekhar–Ambarzumian equation for the H function [2]:

$$H(\mu) = 1+\frac{1}{2}\lambda\mu H(\mu)\int_0^1 \frac{H(\mu')}{\mu+\mu'}\,\mathrm{d}\mu'. \tag{26}$$

If $\tau = 0$ then Eq. (26) follows from Eq. (25) as it should.

Analogically we obtain equation for the g function:

$$g(\tau,\mu) = \mathrm{e}^{-\tau/\mu}+\frac{1}{2}\lambda\mu\int_0^1 \frac{g(\tau,\mu')-g(\tau,\mu)}{\mu'-\mu}H(\mu')\,\mathrm{d}\mu'. \tag{27}$$

4. Approximate formulas for the resolvent

We can find the approximate formula for the resolvent from Eq. (5) by using Eq. (7). After some tedious but straightforward calculations we find for the non conservative case ($\lambda \neq 1$) that

$$\Gamma(\tau,\tau') = \Phi(|\tau-\tau'|)+\sum_{i=1}^{N}\sum_{k=1}^{N}\frac{a_i a_k}{s_i+s_k}[\mathrm{e}^{-s_k|\tau-\tau'|}-\mathrm{e}^{-s_i\tau-s_k\tau'}]. \tag{28}$$

The formula for the conservative case is more complicated:

$$\Gamma(\tau,\tau') = 3\bar{\tau}+\Phi(|\tau-\tau'|)+\sqrt{3}\sum_{k=2}^{N}a_k s_k^{-1}(1-\mathrm{e}^{-s_k\bar{\tau}})[1+\mathrm{e}^{-s_k|\tau-\tau'|}]+\sum_{i=2}^{N}\sum_{k=2}^{N}\frac{a_i a_k}{s_i+s_k}[\mathrm{e}^{-s_k|\tau-\tau'|}-\mathrm{e}^{-s_i\tau-s_k\tau'}]. \tag{29}$$

Using h and g functions we can write Eqs. (28) and (29) in a simpler form:

$$\Gamma(\tau,\tau') = \sum_{i=1}^{N}a_i H(s_i^{-1})\mathrm{e}^{-s_i(\tau'-\tau)}-\sum_{i=1}^{N}a_i[h(\tau,s_i^{-1})-1]\mathrm{e}^{-s_i\tau'}$$

and

$$\Gamma(\tau,\tau') = \sqrt{3}g(\tau,\infty)+\sum_{i=2}^{N}a_i H(s_i^{-1})\mathrm{e}^{-s_i(\tau'-\tau)}-\sum_{i=2}^{N}a_i[h(\tau,s_i^{-1})-1]\mathrm{e}^{-s_i\tau'}.$$

Having obtained an explicit formula for the resolvent we can find the radiation field in a semi-infinite homogeneous isotropically scattering atmosphere with arbitrarily distributed internal sources by integrating Eq. (3).

5. Exponential sources

Now we have all the necessary tools to approach the problem of determination of the radiation field. First, we consider the so-called standard problem, i.e. the case of exponentially distributed sources in the atmosphere. This situation is usually caused by a parallel beam incident on the atmosphere.

We assume that the internal sources are described by the formula

$$B_0(\tau) = B_0\mathrm{e}^{-\tau/\kappa},$$

where κ is a constant. In the case of an incident beam κ is the cosine of the angle of incidence. Sobolev [7] has shown that for this case

$$B(\tau) = B_0 H(\kappa)\left[\mathrm{e}^{-\tau/\kappa} + \int_0^\tau \Phi(t)\mathrm{e}^{-(\tau-t)/\kappa}\,\mathrm{d}t\right]. \tag{30}$$

Taking into account Eq. (10) we have

$$B(\tau) = B_0 H(\kappa) g(\tau, \kappa). \tag{31}$$

Since the diffuse intensities of the radiation moving upward and downward in the atmosphere are expressed in the form, respectively,

$$I(\tau, -\mu) = \int_\tau^\infty B(t)\mathrm{e}^{-(t-\tau)/\mu}\,\mathrm{d}t/\mu, \tag{32}$$

$$I(\tau, \mu) = \int_0^\tau B(t)\mathrm{e}^{-(\tau-t)/\mu}\,\mathrm{d}t/\mu, \tag{33}$$

then using Eqs. (9), (10) and (20) we have

$$I(\tau, -\mu) = \frac{B_0 \kappa H(\kappa)}{\kappa + \mu}[g(\tau, \kappa) + h(\tau, \mu) - 1], \tag{34}$$

$$I(\tau, \mu) = \frac{B_0 \kappa H(\kappa)}{\kappa - \mu}[g(\tau, \kappa) - g(\tau, \mu)]. \tag{35}$$

Here, argument μ is the cosine of the angle between the direction of photon's flight and the positive direction of the τ-axis.

The apparent singularity in Eq. (35) may easily be removed by the L'Hospitale rule:

$$I(\tau, \mu) = B_0 \kappa H(\kappa)\frac{\partial g(\tau, \kappa)}{\partial \kappa},$$

where according to Eq. (14)

$$\frac{\partial g(\tau, \kappa)}{\partial \kappa} = \mathrm{e}^{-\tau/\kappa}\left[\frac{\tau}{\kappa^2} - \sum_{i=1}^N \frac{a_i}{(1 - s_i\kappa)^2} - \frac{\tau}{\kappa}\sum_{i=1}^N \frac{a_i}{1 - s_i\kappa}\right] + \sum_{i=1}^N \frac{a_i \mathrm{e}^{-s_i\tau}}{(1 - s_i\kappa)^2}.$$

It can be seen from Eqs. (34) and (35) that in the conservative case ($\lambda = 1$) the atmosphere is saturated by photons if $\tau \to \infty$ and the radiation field does not depend on the angle $\arccos\mu$ any more. Using the results of Chandrasekhar [2] we have found that in conservative case $a_1 = \sqrt{3}$ and the asymptotic radiation field is described by the formula

$$I_{\mathrm{asympt}} \simeq B_0 \kappa \sqrt{3} H(\kappa), \quad -1 \leqslant \mu \leqslant 1.$$

We may note that in the case of an incident beam

$$B_0 = \tfrac{1}{4}\lambda F,$$

where πF is the net flux of the incident beam per unit area normal to the beam.

6. Milne problem

Next we consider the Milne problem where $B_0(\tau) = 0$ and the source function satisfies the integral equation

$$B(\tau) = \frac{1}{2}\lambda \int_0^\infty E_1(\tau - t)B(t)\,\mathrm{d}t.$$

We understand that there are infinitely powerful sources of radiation at $\tau \to \infty$. In this case the source function can be expressed in the form [6]

$$B(\tau) = B(0)\left[\mathrm{e}^{k\tau} + \int_0^\tau \mathrm{e}^{k(\tau-t)}\Phi(t)\,\mathrm{d}t\right] = B(0)g\left(\tau, -\frac{1}{k}\right), \tag{36}$$

where k is the solution of the characteristic equation

$$1 - \frac{\lambda}{2k}\ln\frac{1+k}{1-k} = 0. \tag{37}$$

It is clear that for the Milne problem we have to normalize the parameters of the radiation field. We choose the source function at $\tau = 0$ to be equal to unity—$B(0) = 1$. Using Eq. (7) in (36) we have for the conservative case

$$B(\tau) = \sqrt{3}[\tau + q(\tau)], \tag{38}$$

where $q(\tau)$ is the Hopf function which in our approximation can be expressed as

$$q(\tau) = \frac{1}{\sqrt{3}}\left[1 + \sum_{i=2}^{N} a_i s_i^{-1}(1 - \mathrm{e}^{-s_i\tau})\right]. \tag{39}$$

In the radiative theory the value of the Hopf function at infinity has played a very important role. In our approximation it becomes

$$q(\infty) = \frac{1}{\sqrt{3}}\left(1 + \sum_{i=2}^{N} a_i s_i^{-1}\right). \tag{40}$$

According to Eqs. (32) and (33) for the conservative case

$$I(\tau, -\mu) = \sqrt{3}[\tau + q(\tau)] + h(\tau, \mu) - 1, \tag{41}$$

$$I(\tau, \mu) = \sqrt{3}[\tau + q(\tau)] - g(\tau, \mu). \tag{42}$$

For the non-conservative case we have

$$B(\tau) = H\left(\frac{1}{k}\right)\mathrm{e}^{k\tau} - h\left(\tau, \frac{1}{k}\right) + 1, \tag{43}$$

$$I(\tau, -\mu) = \frac{1}{1 - k\mu}\left[H\left(\frac{1}{k}\right)\mathrm{e}^{k\tau} - h\left(\tau, \frac{1}{k}\right) + h(\tau, \mu)\right], \tag{44}$$

$$I(\tau, \mu) = \frac{1}{1 + k\mu}\left[H\left(\frac{1}{k}\right)\mathrm{e}^{k\tau} - h\left(\tau, \frac{1}{k}\right) + 1 - g(\tau, \mu)\right]. \tag{45}$$

7. Polynomial sources

Our approximation allows to consider also the polynomial internal sources

$$B_0(\tau) = p_0 + p_1\tau + p_2\tau^2 + \cdots + p_m\tau^m. \tag{46}$$

In this case only the non-conservative atmosphere can be examined since for $\lambda = 1$ the photons cannot escape from the semi-infinite conservative atmosphere quickly enough to maintain a stable radiation field. Defining

$$A_m(\tau) = \int_0^\infty \Gamma(\tau, t)t^m\,\mathrm{d}t, \tag{47}$$

we have from Eq. (4) that

$$B(\tau) = p_0 + p_1\tau + p_2\tau^2 + \cdots + p_m\tau^m + p_0A_0(\tau) + p_1A_1(\tau) + \cdots + p_mA_m(\tau). \tag{48}$$

Taking into account Eq. (4) and using integration by parts we get a differential equation for defining $A_m(\tau)$

$$\frac{\partial A_m}{\partial \tau} = \Phi(\tau)A_m(0) + mA_{m-1}(\tau), \tag{49}$$

since

$$A_m(0) = \int_0^\infty \Phi(t)t^m\,\mathrm{d}t. \tag{50}$$

After having solved Eq. (49) we obtain a recurrent formula to determine $A_m(\tau)$:

$$A_m(\tau) = A_m(0)g(\tau,\infty) + m\int_0^\tau A_{m-1}(t)\,\mathrm{d}t. \tag{51}$$

By direct integration we obtain, using Eq. (50) and setting $A_{-1}(\tau) = 0$,

$$A_0(\tau) = H(\infty)g(\tau,\infty) - 1,$$

where

$$H(\infty) = 1 + \sum_{i=1}^{N} a_i s_i^{-1} = (1-\lambda)^{-1/2}.$$

In order to keep the formulas shorter we examine the internal sources separately. If the internal source is constant—$B_0(\tau) = p_0$—then we have

$$B(\tau) = p_0 H(\infty)g(\tau,\infty), \tag{52}$$

$$I(\tau,-\mu) = p_0 H(\infty)[g(\tau,\infty) + h(\tau,\mu) - 1], \tag{53}$$

$$I(\tau,\mu) = p_0 H(\infty)[g(\tau,\infty) - g(\tau,\mu)]. \tag{54}$$

If the internal source depends on the optical depth linearly then

$$p_1^{-1}B(\tau) = \left\{\kappa^2 \frac{\partial}{\partial\kappa}[H(\kappa)g(\tau,\kappa)]\right\}_{\kappa\to\infty} \tag{55}$$

or

$$p_1^{-1}B(\tau) = \tau H^2(\infty) + g(\tau,\infty)R_2 + H(\infty)[Q_2(\tau) - R_2], \tag{56}$$

where

$$R_m = \sum_{i=1}^{N} a_i s_i^{-m}$$

and

$$Q_m(\tau) = \sum_{i=1}^{N} a_i s_i^{-m}\mathrm{e}^{-s_i\tau}.$$

According to Eqs. (32) and (33) the formulas for the intensities become

$$p_1^{-1}I(\tau,-\mu) = H^2(\infty)(\tau+\mu) + R_2[g(\tau,\infty) + h(\tau,\mu) - 1] - H(\infty)R_2 + H(\infty)\sum_{i=1}^{N}\frac{a_i s_i^{-2}e^{-s_i\tau}}{1+s_i\mu}, \tag{57}$$

$$p_1^{-1} I(\tau,\mu) = H^2(\infty)(\tau-\mu) + R_2[g(\tau,\infty) - g(\tau,\mu)] \\ - H(\infty) R_2 (1-\mathrm{e}^{-\tau/\mu}) + H(\infty) \sum_{i=1}^{N} \frac{a_i s_i^{-2}}{1 - s_i\mu} [\mathrm{e}^{-s_i\tau} - \mathrm{e}^{-\tau/\mu}]. \tag{58}$$

Let us examine also the case when the internal source function is a quadratic function of the optical depth $B_0(\tau) = p_2\tau^2$. Then the source function takes the form

$$(2p_2)^{-1} B(\tau) = R_3 g(\tau,\infty) + \frac{1}{2}\tau^2 H^2(\infty) - R_2^2 + R_3 H(\infty) + R_2 Q_2(\tau) - H(\infty) Q_3(\tau). \tag{59}$$

The intensities of the radiation propagating upward and downward are, respectively,

$$p_2^{-1} I(\tau,-\mu) = 2R_3[g(\tau,\infty) + h(\tau,\mu) - 1] - 2R_2^2 + 2H(\infty)R_3 \\ + H^2(\infty)(\tau^2 + 2\mu\tau + 2\mu^2) + 2R_2 \sum_{i=1}^{N} \frac{a_i s_i^{-2} \mathrm{e}^{-s_i\tau}}{1+s_i\mu} - 2H(\infty) \sum_{i=1}^{N} \frac{a_i s_i^{-3} \mathrm{e}^{-s_i\tau}}{1+s_i\mu}, \tag{60}$$

$$p_2^{-1} I(\tau,\mu) = 2R_3[g(\tau,\infty) + h(\tau,\mu) - 1] - 2R_2^2(1-\mathrm{e}^{\tau/\mu}) \\ + 2H(\infty)R_3(1-\mathrm{e}^{\tau/\mu}) + H^2(\infty)[\tau^2 - 2\mu\tau + 2\mu^2(1-\mathrm{e}^{-\tau/\mu})] \\ + 2R_2 \sum_{i=1}^{N} \frac{a_i s_i^{-2}(\mathrm{e}^{-s_i\tau} - \mathrm{e}^{-\tau/\mu})}{1+s_i\mu} - 2H(\infty) \sum_{i=1}^{N} \frac{a_i s_i^{-3}(\mathrm{e}^{-s_i\tau} - \mathrm{e}^{-\tau/\mu})}{1+s_i\mu}. \tag{61}$$

8. Polynomial sources combined with exponential

Next we consider the case with the internal sources expressed as

$$B_0(\tau) = \tau^m \mathrm{e}^{-\tau/\kappa}, \quad m = 1, 2, 3, \ldots .$$

According to Eq. (3) for the source function we obtain

$$B(\tau) = \tau^m \mathrm{e}^{-\tau/\kappa} + \int_0^\infty \Gamma(\tau,t) t^m \mathrm{e}^{-t/\kappa}\,\mathrm{d}t. \tag{62}$$

Integrating by parts and using Eq. (4) we find that

$$C_m(\tau) = C_m(0) g(\tau,\kappa) + m \int_0^\tau C_{m-1}(t) \mathrm{e}^{-(\tau-t)/\kappa}\,\mathrm{d}t, \tag{63}$$

where

$$C_m(\tau) = \int_0^\infty \Gamma(\tau,t) t^m \mathrm{e}^{-t/\kappa}\,\mathrm{d}t$$

and in our approximation

$$C_m(0) = m!\kappa^{m+1} \sum_{i=1}^{N} \frac{a_i}{(1+s_1\kappa)^{m+1}}.$$

We give the explicit formulas for calculating the radiation field only for the case $m = 1$. Using Eqs. (31), (34), (35), (62) and (63) we obtain for the source function that

$$B(\tau) = \kappa^2 \frac{\partial[g(\tau,\kappa)H(\kappa)]}{\partial\kappa} \tag{64}$$

and for the intensities that

$$I(\tau,-\mu) = \kappa^2 \frac{\partial}{\partial\kappa} \left\{ \frac{\kappa H(\kappa)}{\kappa+\mu} [g(\tau,\kappa) + h(\tau,\mu) - 1] \right\}, \tag{65}$$

$$I(\tau,\mu)=\kappa^2\frac{\partial}{\partial\kappa}\left\{\frac{\kappa H(\kappa)}{\kappa-\mu}[g(\tau,\kappa)-g(\tau,\mu)]\right\}. \tag{66}$$

It is easy to see that if $\kappa\to\infty$ then Eqs. (64)–(66) reduce to Eqs. (55), (57) and (58) where $p_1=1$. If $\kappa=\mu$ the apparent singularity may again be removed by the L'Hospitale rule and in this case

$$I(\tau,\mu)=\kappa^2\left[H(\kappa)\frac{\partial g(\tau,\kappa)}{\partial\kappa}+\kappa\frac{\mathrm{d}H(\kappa)}{\mathrm{d}\kappa}\frac{\partial g(\tau,\kappa)}{\partial\kappa}+\frac{1}{2}\kappa H(\kappa)\frac{\partial^2 g(\tau,\kappa)}{\partial\kappa^2}\right].$$

Here,

$$\frac{\partial^2 g(\tau,\kappa)}{\partial\kappa^2}=\frac{\tau}{\kappa^2}\frac{\partial g(\tau,\kappa)}{\partial\kappa}+\frac{\tau}{\kappa}\mathrm{e}^{-\tau/\kappa}\left[-\frac{2}{\kappa^2}-\frac{2\kappa}{\tau}\sum_{i=1}^{N}\frac{a_is_i}{(1-s_i\kappa)^3}+\frac{1}{\kappa}\sum_{i=1}^{N}\frac{a_i}{1-s_i\kappa}-\sum_{i=1}^{N}\frac{a_is_i}{(1-s_i\kappa)^2}\right]+2\sum_{i=1}^{N}\frac{a_is_i\mathrm{e}^{-s_i\tau}}{(1-s_i\kappa)^3}.$$

9. Infinitesimally thin emitting layer

Let us assume that there is an infinitesimally thin emitting layer at $\tau=\tau_1$ in the atmosphere. According to Eq. (3) the source function is then

$$B(\tau,\tau_1)=\delta(\tau-\tau_1)+\Gamma(\tau,\tau_1), \tag{67}$$

where δ is the Dirac function. If $\tau\leqslant\tau_1$ then the intensities for the nonconservative case are, respectively,

$$\begin{aligned}I(\tau,-\mu,\tau_1)={}&\frac{1}{\mu}h(\tau_1,\mu)\mathrm{e}^{-(\tau_1-\tau)/\mu}+\sum_{i=1}^{N}\frac{a_iH(1/s_i)}{1-s_i\mu}[\mathrm{e}^{-s_i(\tau_1-\tau)}-\mathrm{e}^{-(\tau_1-\tau)/\mu}]\\&-\sum_{i=1}^{N}\frac{a_i\mathrm{e}^{-s_i\tau_1}}{1-s_i\mu}[h(\tau,1/s_i)-h(\tau,\mu)]+\mathrm{e}^{-(\tau_1-\tau)/\mu}\sum_{i=1}^{N}\frac{a_i\mathrm{e}^{-s_i\tau_1}}{1-s_i\mu}[h(\tau_1,1/s_i)-h(\tau_1,\mu)]\\&+\mathrm{e}^{-(\tau_1-\tau)/\mu}\sum_{i=1}^{N}\frac{a_iH(1/s_i)}{1+s_i\mu}-[h(\tau_1,1/s_i)-h(\tau,\mu)]+\mathrm{e}^{-(\tau_1-\tau)/\mu}\sum_{i=1}^{N}\frac{a_ih(\tau_1,1/s_i)}{1+s_i\mu}\mathrm{e}^{-s_i\tau_1},\end{aligned} \tag{68}$$

$$I(\tau,\mu,\tau_1)=\sum_{i=1}^{N}\frac{a_iH(1/s_i)}{1-s_i\mu}\mathrm{e}^{-s_i(\tau_1-\tau)}-\sum_{i=1}^{N}\frac{a_i\mathrm{e}^{-s_i\tau_1}}{1+s_i\mu}[h(\tau,1/s_i)+g(\tau,\mu)-1]. \tag{69}$$

If $\tau\geqslant\tau_1$ then the formulas for the intensities are

$$I(\tau,-\mu,\tau_1)=\sum_{i=1}^{N}\frac{a_iH(1/s_i)}{1+s_i\mu}\mathrm{e}^{-s_i(\tau-\tau_1)}-\sum_{i=1}^{N}\frac{a_i\mathrm{e}^{-s_i\tau}}{1+s_i\mu}[h(\tau_1,1/s_i)-1], \tag{70}$$

$$\begin{aligned}I(\tau,\mu,\tau_1)={}&\frac{1}{\mu}\mathrm{e}^{-(\tau-\tau_1)/\mu}+\mathrm{e}^{-(\tau_1-\tau)/\mu}\sum_{i=1}^{N}\frac{a_iH(1/s_i)}{1+s_i\mu}-\mathrm{e}^{-(\tau_1-\tau)/\mu}\sum_{i=1}^{N}\frac{a_i\mathrm{e}^{-s_i\tau_1}}{1+s_i\mu}[h(\tau_1,1/s_i)+g(\tau,\mu)-1]\\&+\sum_{i=1}^{N}\frac{a_iH(1/s_i)}{1-s_i\mu}[\mathrm{e}^{-s_i(\tau-\tau_1)}-\mathrm{e}^{-(\tau-\tau_1)/\mu}]-\sum_{i=1}^{N}\frac{a_i}{1-s_i\mu}[h(\tau_1,1/s_i)-1][\mathrm{e}^{-s_i\tau}-\mathrm{e}^{-s_i\tau_1-(\tau-\tau_1)/\mu}].\end{aligned} \tag{71}$$

10. Some illustrations and the accuracy of the method

We give some examples of numerical calculations by the described method in Figs. 1–7. For all the cases discussed in this paper we have tried to present the dependence of the intensities on the optical depth. We see that there is a discontinuity in intensities at $\mu=0$ and at $\tau=0$ since we require that there is no radiation

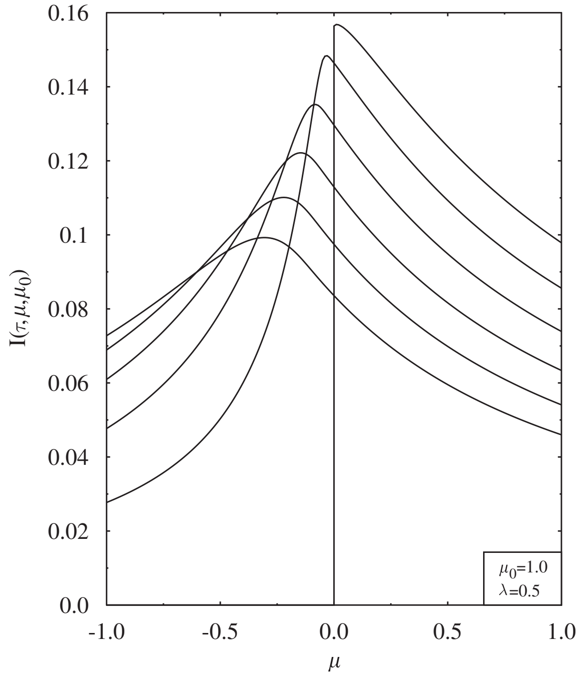

Fig. 1. Intensities for the case of exponentially distributed internal sources. Right side, from above: $\tau = 0.0$; 0.1; 0.2; 0.3; 0.4; 0.5.

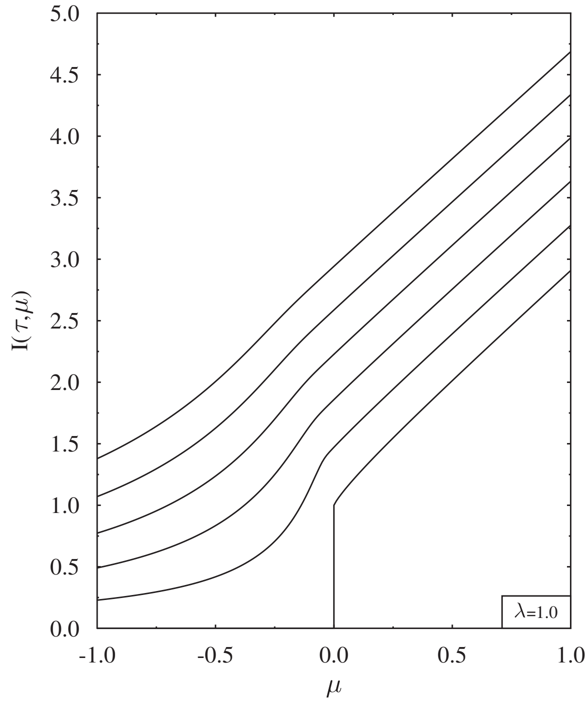

Fig. 2. Same as in Fig. 1 for the Milne problem.

incident on the boundary of the atmosphere. For the case with a radiating surface it is clear that at the depth of that surface and at $\mu = 0$ the numerical value of the intensity theoretically approaches infinity but this behaviour could not be depicted in full in Fig. 7.

In the Appendix we have given the expressions of the source functions and the intensities of the emerging radiation for some theoretical internal source functions just in order to demonstrate the possibilities of the kernel approximation method.

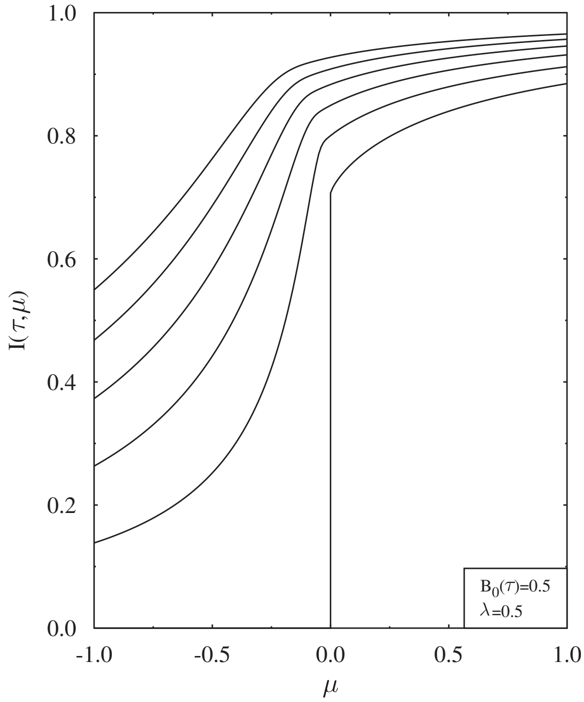

Fig. 3. Same as in Fig. 1 for the case of constant sources.

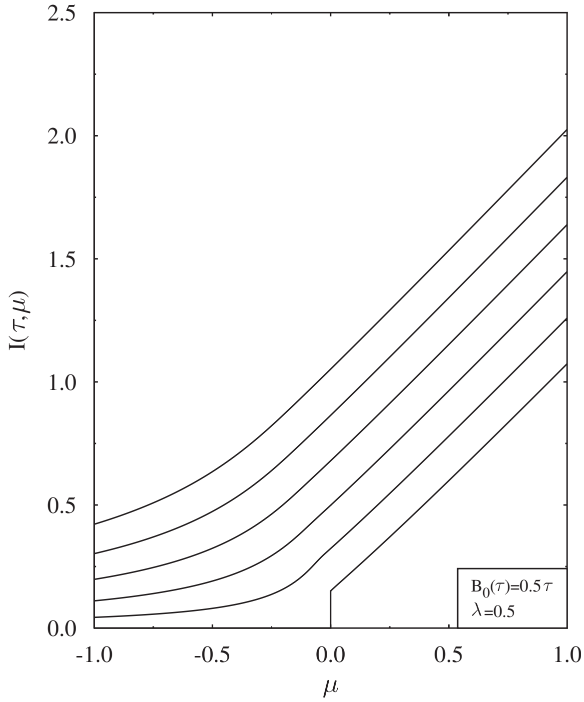

Fig. 4. Same as in Fig. 1 for the case of linear internal sources.

The resolvent function $\Phi(\tau)$ has a logarithmic singularity at $\tau = 0$—this behaviour follows from Eq. (6) (the properties of the resolvent function are perhaps best described by Ivanov [6]).

Accordingly, Eq. (7) can never approximate this behaviour well enough. Fortunately, the resolvent function is usually not the function we are interested in, at least in the sense of its numerical values. What we are traditionally interested in are the intensities and flux. As we have seen, these functions can be expressed as certain integrals over optical depth where the resolvent function is weighted by some exponential function.

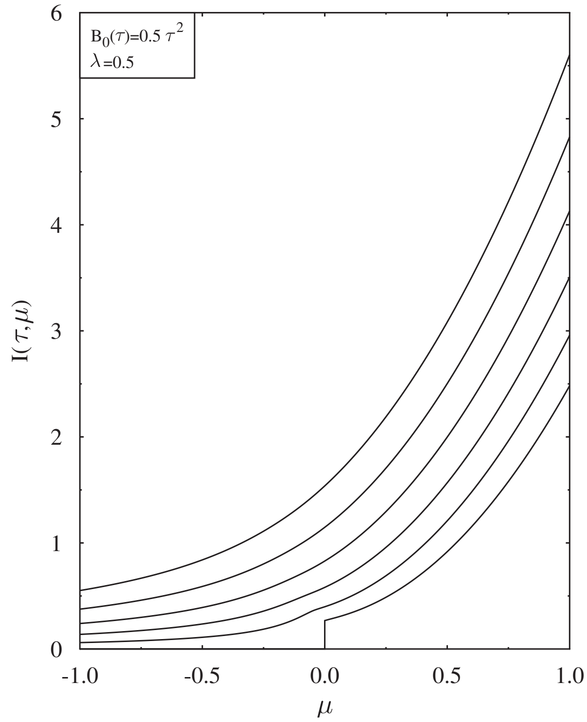

Fig. 5. Same as in Fig. 1 for the case of quadratic internal sources.

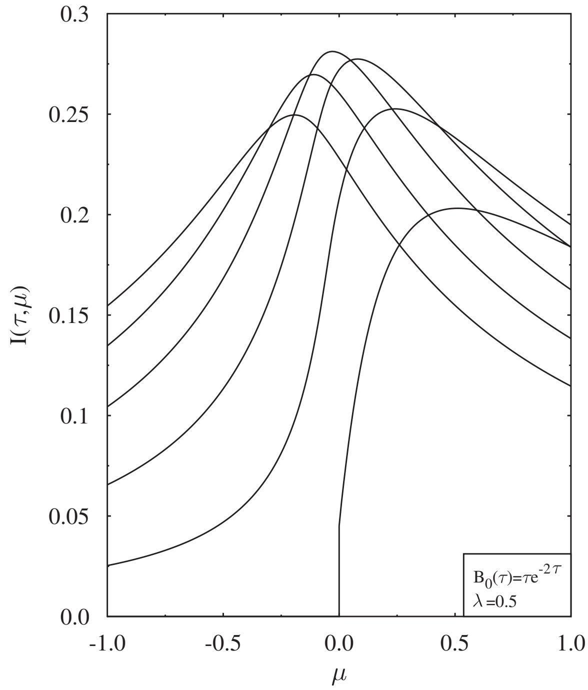

Fig. 6. Same as in Fig. 1 for the case of $B_0(\tau) = \tau e^{-\tau/\kappa}$.

This means that the singularity is smeared over a range of optical depths thus effectively eliminating the influence of the singularity in the final results.

The calculations have shown that the accuracy of the method decreases quite substantially toward smaller values of the angular variable and slightly toward larger values of λ. However, the loss of accuracy at small values of μ is not very serious, as has been pointed out by Bosma and De Rooij [14] since the range of μ values

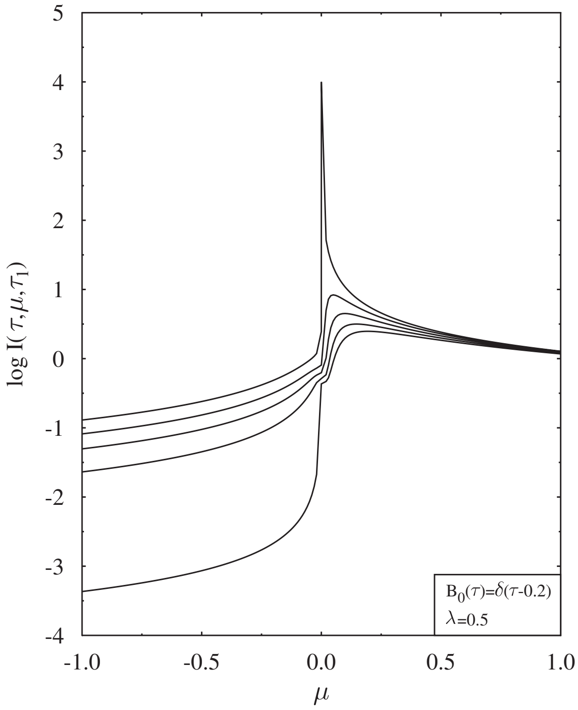

Fig. 7. Same as in Fig. 1 for the case of infinitesimally thin layer of constant sources.

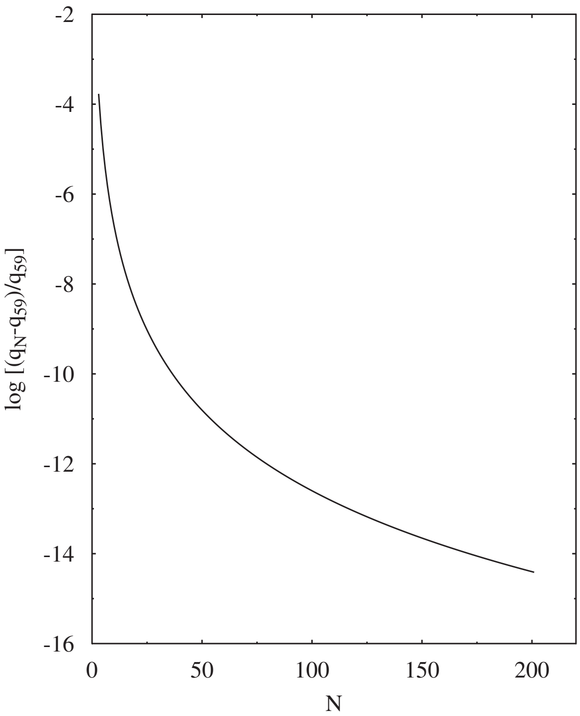

Fig. 8. The relative accuracy of the $q_N(\infty)$ as a function of the Gaussian order N.

where the less accurate results occur is rather limited and the loss of accuracy does not influence the moment values which are generally more important in practical calculations.

There are different ways to check the accuracy of the method. One can choose from a wealth of equations and then compare the respective LHS and RHS. In a paper [15] one of the authors (T.V.) has pointed out that the formulas which contain integrals over the angular variable tend to give essentially more accurate results than those without. So we are left with a crude but still reliable method—we gradually increase the number of quadrature points and observe the appearance of—supposedly—correct significant figures. Still, one of the pet methods to check the accuracy in radiative transfer calculations is to use the H function. We took for standard the calculations of Bosma and De Rooij [14] which are perhaps the best analysed results for the H function. In getting their results they used a modified form of Eq. (26) and the Gauss–Legendre quadrature rule with $N = 128$. This allowed to obtain the accuracy $\varepsilon \leqslant 10^{-12}$. When using Eq. (13) we obtained the same accuracy at $N = 61$. For illustration we present Fig. 8 where the accuracy of the Hopf function at infinity is plotted as a function of the order of the Gauss–Legendre quadrature. The value of $q(\infty)$ with the accuracy of 10^{-59} is given in [15].

The described approximation can easily be generalized for a finite homogeneous isotropically scattering atmosphere [13,15]. The Chandrasekhar pseudoproblems render themselves to this approximation, too. Moreover, one of the authors (R.R.) has applied this approximation to homogeneous anisotropically scattering atmospheres by using the Sobolev approach [11].

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Appendix A

Formulas for the source functions and the intensities of the emerging radiation for some internal source functions:

No.	$B_0(\tau)$	$B(\tau)$	$I(0,\mu)$
1	$E_1(\tau)$	$(2/\lambda)\Phi(\tau)$	$(2/\lambda\mu)[H(\mu)-1]$
2	$E_2(\tau)$	$(2/\lambda)[1-\sqrt{1-\lambda}g(\tau,\infty)]$	$H(\mu)(\alpha_0-2/\lambda)+2/\lambda$
3	$E_n(\tau)$	$\int_0^1 H(\mu)g(\tau,\mu)\mu^{n-2}\,\mathrm{d}\mu$	$H(\mu)\int_0^1 s^{n-1}H(s)(\mu+s)^{-1}\,\mathrm{d}s$
4	$\Phi(\tau)$	$\lambda\partial\Phi(\tau)/\partial\lambda$	$(\lambda/\mu)\partial H(\mu)/\partial\lambda$

Here α_0 is the zeroth moment of the H function.

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