Buoyancy Wave Interaction with Critical Levels in the Atmosphere

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Abstract

General non-linear internal buoyancy wave equation (BWE) is developed, which has the second (differential) order in space but the third order in time. The wave operator splitting method is then applied to the stationary BWE to get the orographically generated buoyancy waves in the thermally stratified atmosphere for altitude-variable wind conditions. The splitting method is further generalized to a critical level (CL) containing atmosphere. The CLs, which appear in the atmosphere (including the planetary boundary layer) if the wind weakens and changes direction or rotates with the altitude, will provide a break-up of the atmosphere to the regular (for BWE) layers separated by singular critical levels at which the differential order of the BWE is lowered.
This lowering will cause either partial or full reflection of waves, though in the special fast wind altering case, the CL can prove transparent, too. The wave modelling examples for different CLs are provided.
Classification of the wind situations with respect to the various reflection-transparency types of CL is vital for wave stress (vertical flux of mean horizontal momentum) and upper-level wave breaking study.

1. General BWE

$$\frac{d_0}{dt}\left(\frac{d_0^2}{dt^2}+f^2\right)\eta^2\frac{\partial^2 w}{\partial\eta^2}+f\mathcal{R}_1\frac{d_0}{dt}\eta\frac{\partial w}{\partial\eta}-f^2\mathcal{A}_1\eta\frac{\partial w}{\partial\eta}-\mathcal{A}_2\frac{d_0^2 w}{dt^2}+$$

$$+\left[\left(\frac{d_0^2}{dt^2}+N^2\right)H^2\nabla^2+f\left(\mathcal{R}_2-\mathcal{R}_1\right)\right]\frac{d_0}{dt}w-f\mathcal{A}_1\mathcal{R}_1 w=\tilde{S}$$

$$\eta=p/p_s \qquad w=\dot\eta\equiv\frac{d\eta}{dt} \qquad \frac{d_0}{dt}=\frac{\partial}{\partial t}+\mathbf{U}(\eta)\cdot\nabla$$

2. Spectral stationary BWE

$$w(\mathbf{x},\eta,t)=\sum_{\nu,\mathbf{k}}\widehat{w}_{\mathbf{k}}^{\nu}(\eta)e^{\mathbf{i}[\nu t-(\mathbf{k}\cdot\mathbf{x})]}$$

$$a_{\mathbf{k}}(\eta)\eta^2\frac{\partial^2\widehat{w}_{\mathbf{k}}}{\partial\eta^2}+b_{\mathbf{k}}(\eta)\eta\frac{\partial\widehat{w}_{\mathbf{k}}}{\partial\eta}+c_{\mathbf{k}}(\eta)\widehat{w}_{\mathbf{k}}=0$$

$$a_{\mathbf{k}}(\eta)=\mathbf{k}\cdot\mathbf{U}(\eta)[f^2-(\mathbf{k}\cdot\mathbf{U}(\eta))^2]$$

$$b_{\mathbf{k}}(\eta)=f^2\alpha_{\mathbf{k}}^1(\eta)+\mathbf{i}f[\mathbf{k}\cdot\mathbf{U}(\eta)]\beta_{\mathbf{k}}^1(\eta)$$

$$c_{\mathbf{k}}(\eta)=\mathbf{k}\cdot\mathbf{U}\left[H^2k^2\left((\mathbf{k}\cdot\mathbf{U})^2-N^2\right)-\mathbf{k}\cdot\mathbf{U}\alpha_{\mathbf{k}}^2\right]$$
$$+\mathbf{i}f\left[\alpha_{\mathbf{k}}^1\beta_{\mathbf{k}}^1+\mathbf{U}\cdot\mathbf{k}(\beta_{\mathbf{k}}^2-\beta_{\mathbf{k}}^1)\right]$$

$$\alpha_{\mathbf{k}}^0(\eta)=\mathbf{k}\cdot\mathbf{U}(\eta) \qquad \alpha_{\mathbf{k}}^1=\eta\frac{\partial\alpha_{\mathbf{k}}^0}{\partial\eta} \qquad \alpha_{\mathbf{k}}^2=\eta\frac{\partial\alpha_{\mathbf{k}}^1}{\partial\eta}-\alpha_{\mathbf{k}}^1$$

$$\beta_{\mathbf{k}}^1=\mathbf{e}_z\cdot\left(\eta\frac{\partial\mathbf{U}\times\mathbf{k}}{\partial\eta}\right) \qquad \beta_{\mathbf{k}}^2=\eta\frac{\partial\beta_{\mathbf{k}}^1}{\partial\eta}$$

3. Temperature, Brunt-Väisälä frequency and wind profiles

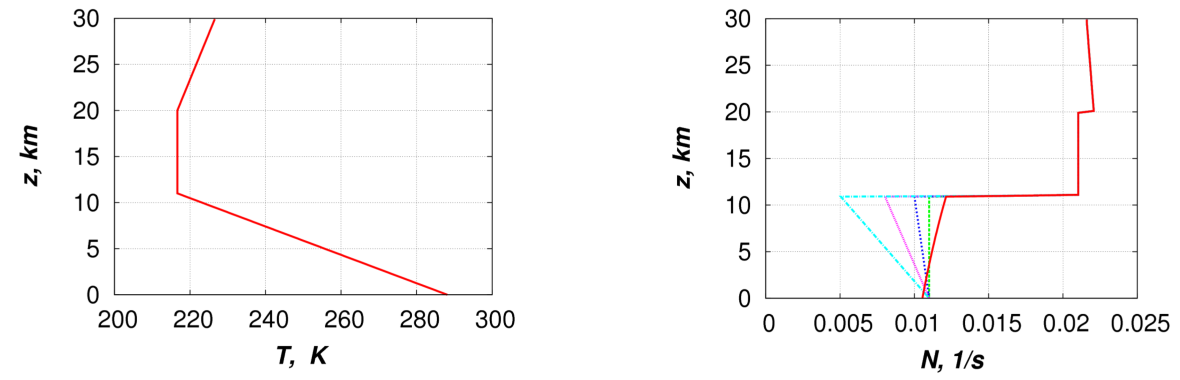


Figure 1. The US'76 standard temperature and consequent buoyancy frequency N (red lines).

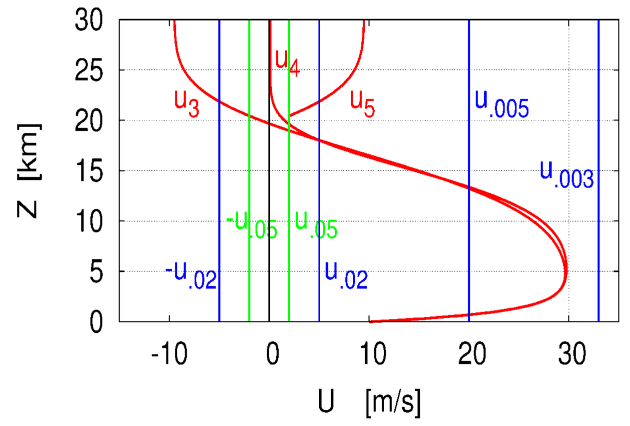
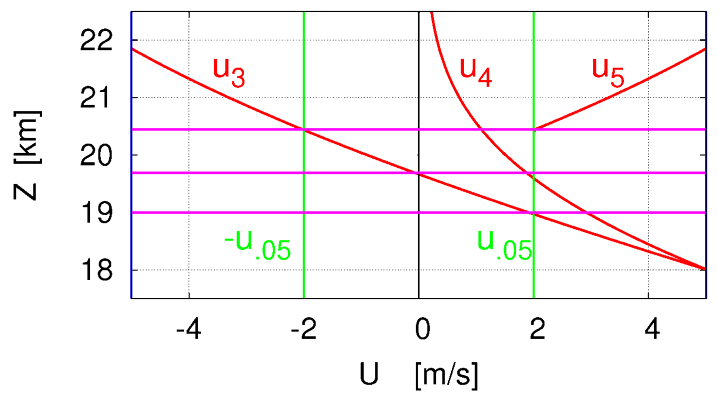


Figure 2. Wind profiles, applied in modelling (U_3,U_4,U_5). Vertical lines are critical velocities Uk = f/k; where f represents the Coriolis paramete and ; k is the wave-number ([k] = 1/km).

4 Layering generated by critical levels

Figure 3. Critical level origin.



Critical levels z_c are those with $a_k(z_c)$ = 0, i.e. whith either $U(z_c)$ = 0, or $|U(z_c)|$ = f/k

Critical levels provide layering of the atmosphere to the hypo- and hypercritical zones. Hypercritical or evanescent layers are those with $|U(z)| < f/k$;
The hypocritical or free-wave layers correspond to $|U(z)| > f/k$.

The order of BWE reduces at the critical level by one degree. The BWE represents there a first order differential equation:

$$\left[b_{\mathbf{k}}(\eta)\eta\frac{\partial\widehat{w}_{\mathbf{k}}}{\partial\eta}+c_{\mathbf{k}}(\eta)\widehat{w}_{\mathbf{k}}\right]_{\eta_c}=0$$

Consequently, the spectral BWE must be solved for each regular hypo- and hypercritical subdomain in separation. Then, the subdomain solutions must be smoothly continued with continuous first derivatives through critical levels to obtain the global solution.

5. BWE factorization in regular subdomain

$$\left(\eta\frac{\partial}{\partial\eta}\right)^2\omega+q(\eta)\left(\eta\frac{\partial}{\partial\eta}\right)\omega+r(\eta)\omega=0$$

$$q(\eta)=b(\eta)/a(\eta)-1 \qquad r(\eta)=c(\eta)/a(\eta)$$

$$\left(\eta\frac{\partial}{\partial\eta}+\frac{q}{2}+\varphi\right)\left(\eta\frac{\partial}{\partial\eta}+\frac{q}{2}-\varphi\right)\omega=0$$

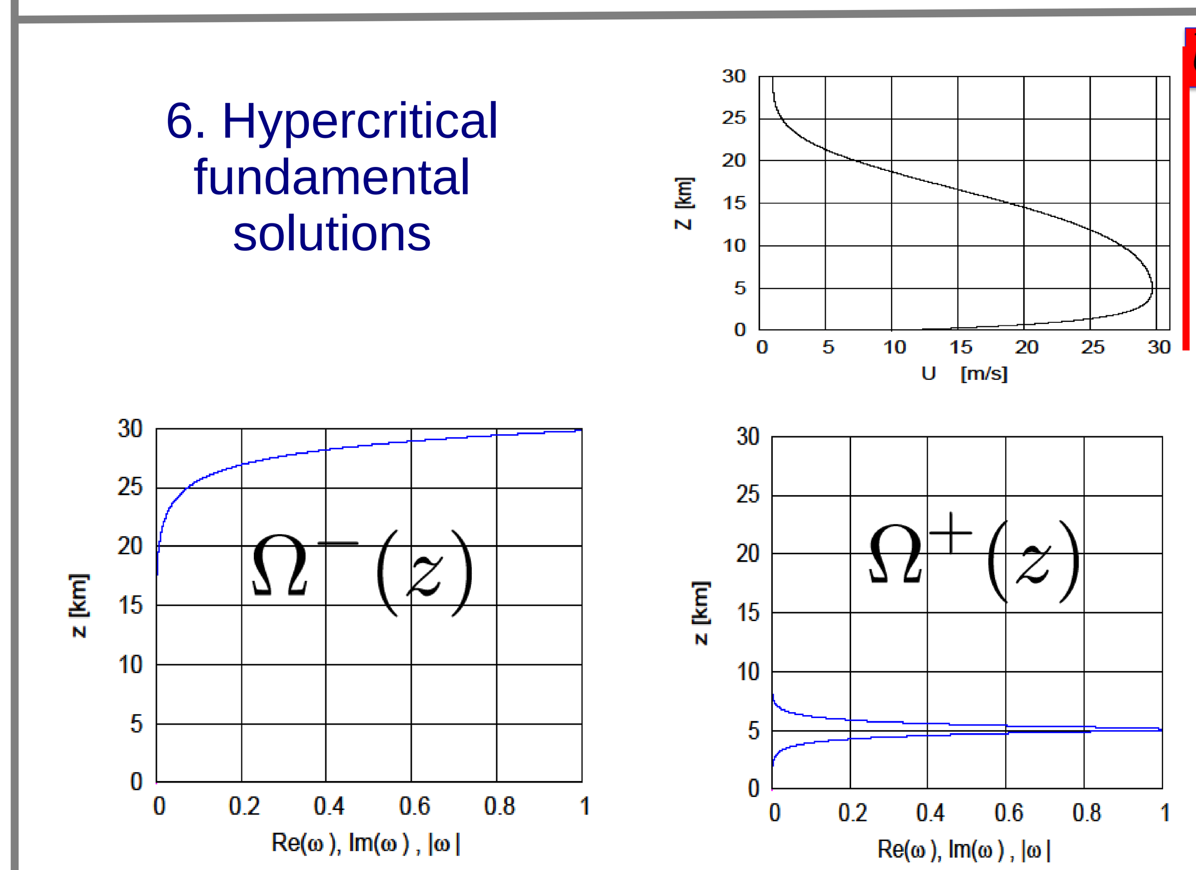
$$\eta\frac{\partial}{\partial\eta}\left(\frac{q}{2}+\varphi\right)-\varphi^2=r-\frac{q^2}{4}$$

$$\left(\eta\frac{\partial}{\partial\eta}+\frac{q}{2}+\varphi\right)\Omega^-=0 \qquad \left(\eta\frac{\partial}{\partial\eta}+\frac{q}{2}-\varphi\right)\Omega^+=0$$

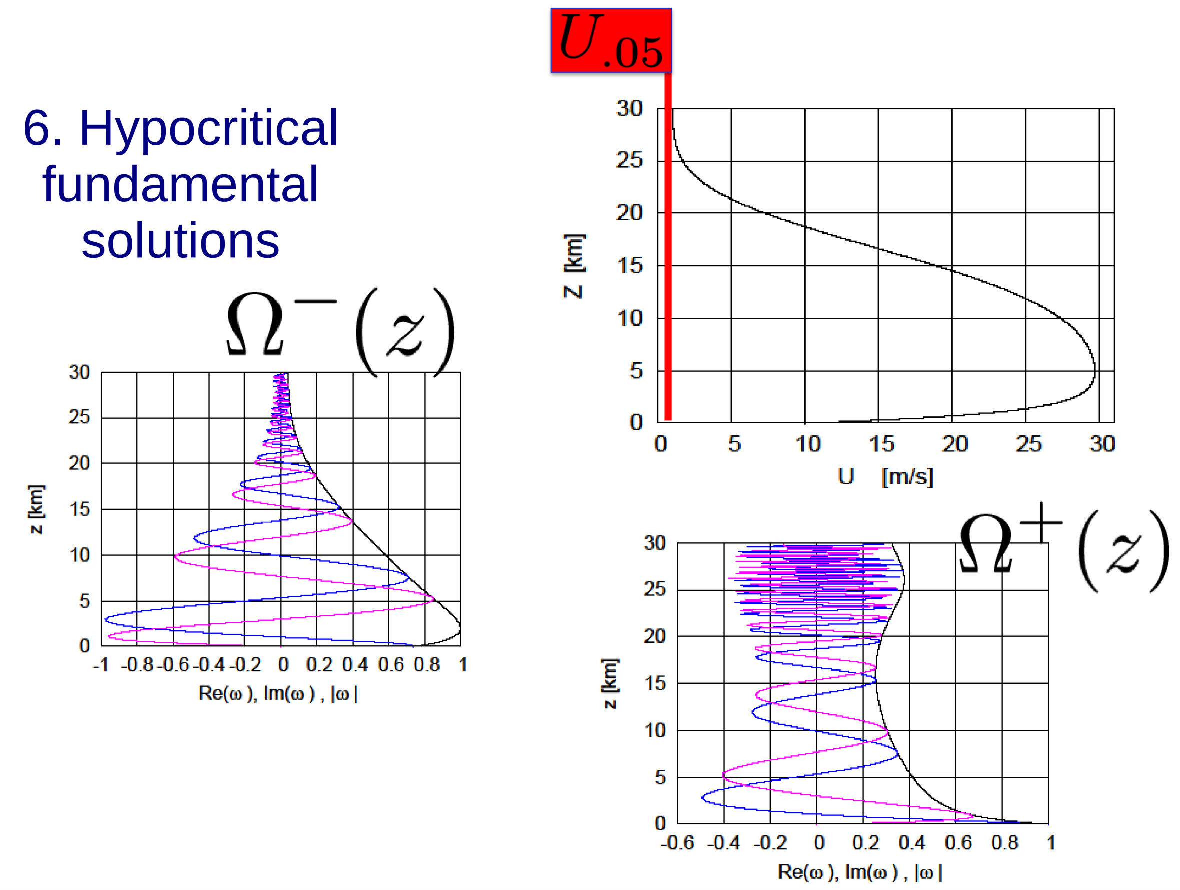
Fundamental solutions Ω^+,Ω^- of these two equations give the general solution in the regular subdomain:

$$w_k(\eta)=c_+\Omega^+(\eta)+c_-\Omega^-(\eta)$$

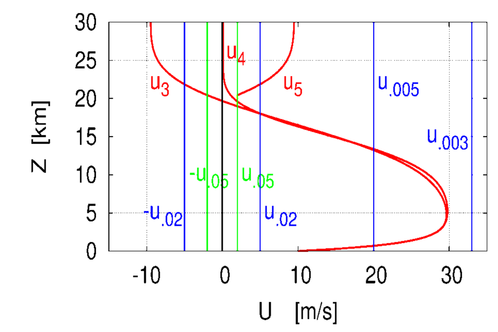
6. Hypercritical fundamental solutions

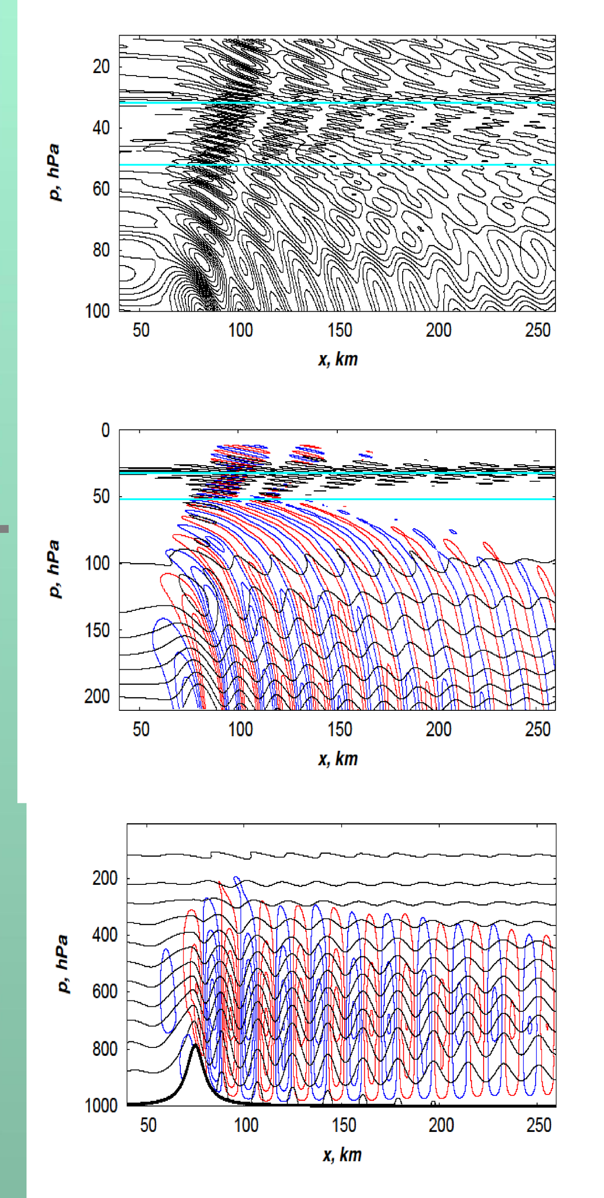


6. Hypocritical fundamental solutions

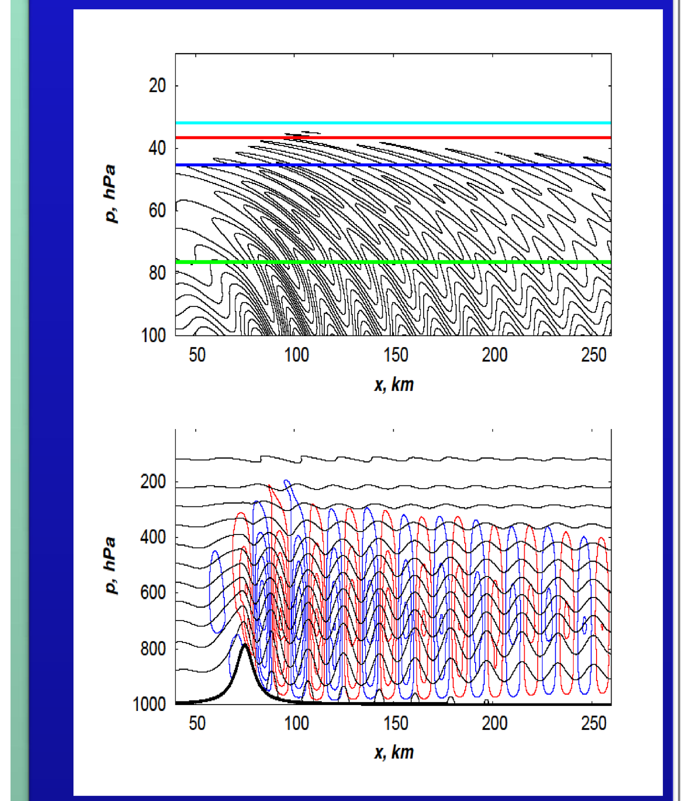


7. BW without critical levels, wind model U_5





8. BW with critical levels, wind model U_4



9. Conclusions

A hypercritical layer on top of a hypocritical one represents an almost ideal absorber independently of its geometrical thickness. At the passing of the critical level, the rising free waves are transformed into evanescent rapidly decreasing waves without any notable reflection on critical level.