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Stream-potential formulation of atmospheric dynamics in pressure coordinates

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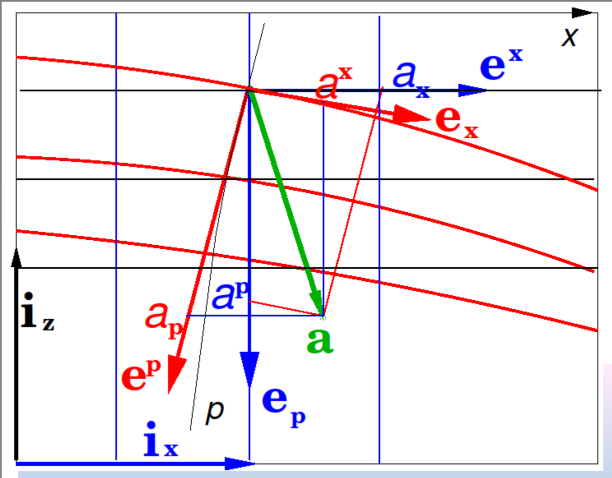
Abstract

Departing from covariant presentation of non-hydrostatic atmospheric dynamics in isobaric coordinates [R. Rõõm 1990: General form of non-hydrostatic equations of atmospheric dynamics in isobaric coordinates. Atm. Ocean Physics. v. 26, 17 - 26. [http://meteo.physic.ut.ee/room/papers/kuni2004/1990.General_Form.pdf] equations of 4-dimensional stream-potential dynamics are derived.

1. Vectors and tensors in pressure coordinates

While passing from ordinary coordinates x,y,z,t to the pressure-coordinate frame *x,y,p(x,y,z,t),t*, a vector with components $\underline{v}^x, \underline{v}^y, \underline{v}^z$, in orthogonal vector basis $\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z$ will have in **p**(ressure)**-space** contravariant components v^x, v^y, v^p in covariant basis $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_p$ and covariant components v_x, v_y, v_p in contravariant basis $\mathbf{e}^x, \mathbf{e}^y, \mathbf{e}^p$

$$\mathbf{V} = \mathbf{i}_\alpha \underline{v}^\alpha = \mathbf{e}_\alpha v^\alpha = \mathbf{e}^\alpha v_\alpha$$



In particular, if V is the velocity of a material particle then

$$\underline{v}^x = dx/dt, \quad v^x = dx/dt$$
$$\underline{v}^y = dy/dt, \quad v^y = dy/dt$$
$$\underline{v}^z = dz/dt, \quad v^p = dp/dt = \omega$$

Properties of curvilinear p(pressure)-space are specified with the metric tensor with $H = RT/g$

$$G^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p^2/H^2 \end{pmatrix}$$

The most important geometrical objects in p-space are the geopotential height gradient, velocity divergence and forticity (velocity curl)

$$\xi^0 = \partial_\alpha v^\alpha$$
$$(\nabla z)_\alpha = \partial_\alpha z$$
$$\xi^j = (\boldsymbol{\nabla} \times \mathbf{v})^j = -\frac{p}{H}\varepsilon^{j\alpha\beta}\partial_\alpha v_\beta, \; v_\beta = G_{\beta\gamma}v^\gamma.$$

2. Velocity 4-potential

$$\{\psi^0, \; \boldsymbol{\psi}\} = \{\psi^0, \psi^x, \psi^y, \psi^p\}$$

Velocity can be presented as the sum of scalar-potential gradient and vector-potential curl :

$$v^j = G^{j\alpha}\partial_\alpha\psi^0 + \varepsilon^{j\alpha\beta}\partial_\alpha\psi_\beta \quad \textbf{(1)}$$

3. Equations for potentials

$$\mathcal{L}^0\psi^0 = \xi^0 \qquad \mathcal{L}\psi^p = -\frac{p^2}{H^2}\xi^p$$
$$\mathcal{L}\omega = \frac{p^2}{H^2}(\partial_y\xi^x - \partial_x\xi^y)$$
$$\nabla_h^2\psi_x = -\partial_x\partial_p\psi^p - \partial_y\omega \quad \textbf{(2)}$$
$$\nabla_h^2\psi_y = \partial_x\omega - \partial_y\partial_p\psi^p$$

Where the Laplacians are

$$\mathcal{L}^0 = \frac{\partial}{\partial p}\frac{p^2}{H^2}\frac{\partial}{\partial p} + \nabla_h^2 \qquad \nabla_h^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$
$$\mathcal{L} = \frac{p^2}{H^2}\frac{\partial^2}{\partial p^2} + \nabla_h^2$$

For use of these elliptic equations, with sources expressed in terms of velocity divergence and curl , evolutional equations for 4-vector are required.

$$\{\xi^0, \; \boldsymbol{\xi}\} = \{\xi^0, \xi^x, \xi^y, \xi^p\}$$

4. Nonhydrostatic equation of motion in p-space

$$\frac{\partial\mathbf{v}}{\partial t} = -\boldsymbol{\Xi}\times\mathbf{v} - \boldsymbol{\nabla}(gz + \mathbf{v}\cdot\mathbf{v}/2) - \mathbf{F}$$
$$\boldsymbol{\Xi} = \mathbf{i}_z f + \boldsymbol{\xi} \qquad (\boldsymbol{\Xi}\times\mathbf{v})^j = -\varepsilon^{j\alpha\beta}\Xi_\alpha v_\beta$$
$$\mathbf{F}: \quad F^x = 0, \; F^y = 0, \; F^p = \omega\partial_t \ln H$$

Application of **div** and **curl** to (3) yields evolutional equations

$$\frac{\partial\xi^0}{\partial t} = -\mathcal{L}^0(gz + \mathbf{v}\cdot\mathbf{v}/2) - \nabla\cdot(\boldsymbol{\Xi}\times\mathbf{v} + \mathbf{F}) \quad \textbf{(3)}$$
$$\frac{\partial\boldsymbol{\xi}}{\partial t} = \frac{H}{p}\boldsymbol{\nabla}\times(\boldsymbol{\Xi}\times\mathbf{v} + \mathbf{F}) \quad \textbf{(4)}$$

5. Lower boundary equation in NH p-coordinate dynamics

Continuity equation for nondimensional density ***n*** : in p-space reads

$$n = -\frac{p}{H}\frac{\partial z}{\partial p} \quad \textbf{(5)}$$
$$\frac{\partial n}{\partial t} = -\xi^0 \quad \textbf{(6)}$$

While mass conservation in a unit column over surface p_s(x,y,t) is

$$\frac{\partial}{\partial t}\int_0^{p_s} n dp = -\nabla\cdot\int_0^{p_s}\mathbf{v}dp$$

For relative surface pressure fluctuation $\chi = p_s'/p_s$

a comparison of this conservation law with vertically integrated Eq. **(6)** yields general lower boundary (free surcase) evolution equation

$$\frac{\partial\chi}{\partial t} = \left(\frac{p}{H^2}\frac{\partial\psi^0}{\partial p}\right)_s - \frac{(\nabla_h\psi^0)_s\cdot\nabla p_s}{p_s} \quad \textbf{(7)}$$

valid both in NH case and at HS limit.

6. Splitting of NH geopotential

$$(5) \Rightarrow z = z_s(x,y) + \int_p^{p_s(x,y,p,t)}(nH)(x,y,p',t)\frac{dp'}{p'}$$

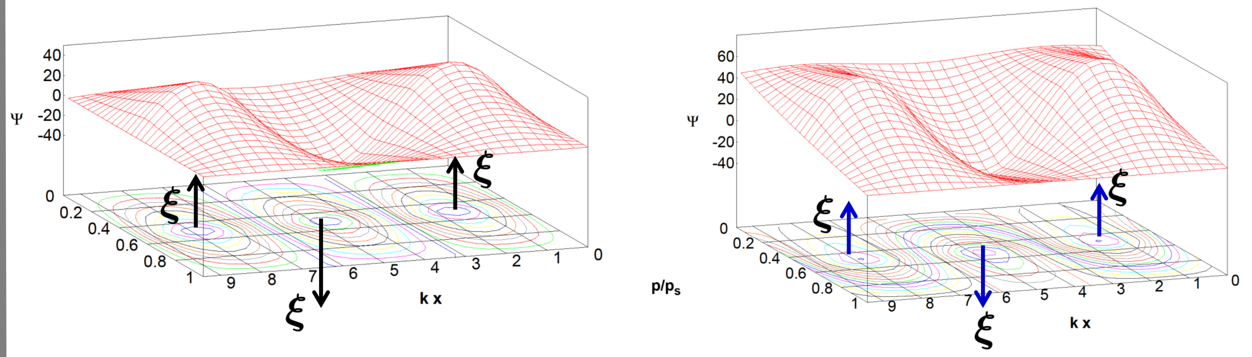
$$gz(x,y,p,t) = \quad \textbf{(8)}$$

Background	$= g\bar{z}(p)$
Thermal component	$+gz^T$, $\quad gz^T = R\int_p^{p_s(x,y,t)} T'(x,y,p',t)\frac{dp'}{p'}$
Barotropic component, rising from surface pressure fluctuation	$+C^2\chi(x,y,t)$
Nonhydrostatic component	$+gz'(x,y,p,t)$

$$C^2 = g\overline{H}_s = R\overline{T}_s$$ Isochoric sound speed on surface

7. Solenoidal subsystem

$$(2), (4) \Rightarrow \frac{\partial\boldsymbol{\xi}}{\partial t} = \frac{H}{p}\boldsymbol{\nabla}\times(\boldsymbol{\Xi}\times\mathbf{v} + \mathbf{F})$$
$$\mathcal{L}\psi^p = -\frac{p^2}{H^2}\xi^p \qquad \mathcal{L}\omega = \frac{p^2}{H^2}(\partial_y\xi^x - \partial_x\xi^y)$$
$$\nabla_h^2\psi_x = -\partial_x\partial_p\psi^p - \partial_y\omega$$
$$\nabla_h^2\psi_y = \partial_x\omega - \partial_y\partial_p\psi^p$$



8. Potential subsystem

From **(2),(3),(8)**, for generalized geopotential $\varphi = gz' + C^2\chi + \frac{\mathbf{v}^2}{2}$

and flow potential ψ^0

two equations proceed

$$\frac{1}{C^2}\frac{\partial^2}{\partial t^2}\left(p\frac{\partial\varphi}{\partial p}\right) + \mathcal{L}^0\varphi = -S^T - S^V$$
$$\mathcal{L}^0\psi^0 - \frac{\partial}{\partial t}\left(p\frac{\partial\varphi}{\partial p}\right) = -S_\psi$$

which describe evolution of potential subsystem: surface pressure evolution and adjustment, external gravity waves, secondary circulations

Boundary conditions at the surface are:

$$\left(\frac{\partial\varphi}{\partial p}\right)_s = 0 \quad \psi_s^0 = 0$$ Usual traditional

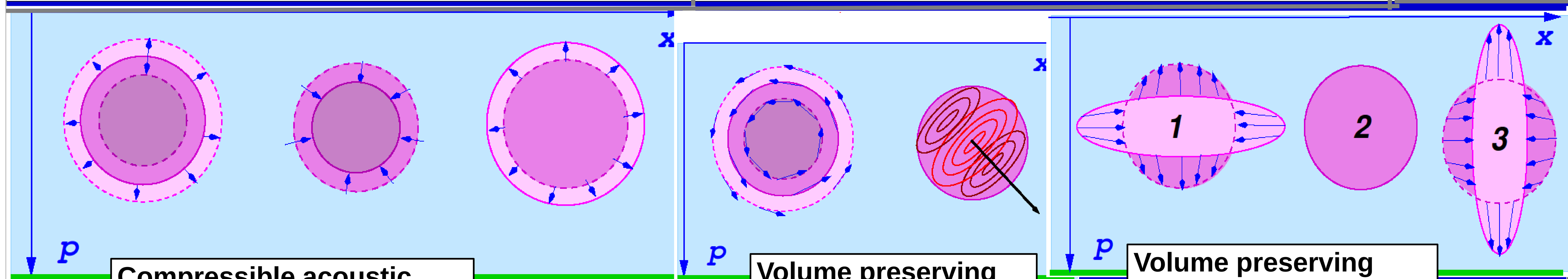
$$\frac{\partial\varphi_s}{\partial t} = \frac{g^2}{C^2}\left(p\frac{\partial\psi^0}{\partial p}\right)_s - S_s$$ New, describes surface pressure evolution, responsible for mass conservation

Sources for potential subsystem

$$S^V = \nabla\cdot(\boldsymbol{\Xi}\times\mathbf{v} + \mathbf{F}) - \frac{p}{2C^2}\frac{\partial}{\partial p}\frac{\partial^2\mathbf{v}^2}{\partial t^2} \qquad S^T = \nabla_h^2 gz^T$$
$$S_\psi = \frac{p}{2C^2}\frac{\partial}{\partial p}\frac{\partial\mathbf{v}^2}{\partial t}$$

are created by solenoidal flow and temperature distribution

$$S_s = \nabla_h\psi_s^0\cdot\nabla_h\chi \qquad \chi = \frac{\varphi_s - \mathbf{v}^2/2}{C^2}$$



Compressible acoustic oscillation

Volume preserving solenoidal rotation

Volume preserving hydrostatic oscillation